

Tensor Product of Polygonal Cell Complexes

Yu-Yen Chien
 Mathematics Division
 National Center for Theoretical Science
 TAIWAN
 yychien@ncts.ntu.edu.tw

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Abstract

We introduce the tensor product of polygonal cell complexes, which interacts nicely with the tensor product of link graphs of complexes. We also develop the unique factorization property of polygonal cell complexes with respect to the tensor product, and study the symmetries of tensor products of polygonal cell complexes.

1 Introduction

A **polygonal cell complex** is a 2-dimensional CW-complex with polygons as 2-cells, namely a graph with polygons attached. To be precise, we take a rather formal definition: a polygonal cell complex is a 2-dimensional CW-complex satisfying:

- (1) Each 1-cell is an interval of length 1, and each 2-cell is a disc of positive integral circumference.
- (2) For a 2-cell of circumference n , the attaching map sends exactly n points evenly distributed on the boundary to the 0-skeleton.
- (3) For each boundary segment between the points described in (2), the attaching map sends the segment isometrically onto an open 1-cell.

Intuitively speaking, we can think of each 2-cell as a regular polygon, and the attaching map glues vertices to vertices, and edges to edges. Those 2-cells act like faces of a polyhedron, and we will use the word face to denote a 2-cell alternatively. Note that the attaching map of a face may not be injective, and a polygonal cell complex can be quite different from polyhedra. A **polygonal complex**, which simulates polyhedra better, is a polygonal cell complex satisfying:

- (i) The attaching map of each cell is injective.
- (ii) The intersection of any two closed cell is either empty or exactly one closed cell.

Unless otherwise specified, when we use the word complex, it means polygonal cell complex, which may or may not be a polygonal complex.

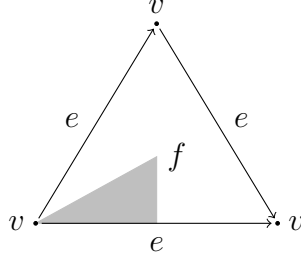


Figure 1: a flag of a dunce hat

Here is a concise way to describe the local structure of complexes. For a polygonal cell complex X , the **link** of X at a vertex v is a graph $L(X, v)$ with vertices indexed by ends of edges attached to v , and edges indexed by corners of faces attached to v . Two vertices v_1 and v_2 in $L(X, v)$ are joined by an edge e if and only if the corresponding ends of v_1 and v_2 are joined by the corresponding corner of e . Basically a link describes the incidence relation of edges and faces at a vertex. Note that $L(X, v)$ can also be identified as the set $\{x \in X \mid d(x, v) = \delta\}$, where d is the distance function in X and δ is some positive number less than $1/2$.

Take the dunce hat in Figure 1 as an example. Although there is only one edge in the complex, this edge has two ends attached to v , and therefore contributes two vertices to the link at v . Notice that the top corner of the face joins these two ends, and corresponds to an edge joining two vertices in the link at v . The left corner of the face joins the same end of the edge, and hence corresponds to a loop in the link, while the right corner of the face also corresponds to a loop at the other vertex. Therefore the link at v is a graph with two vertices e_1 and e_2 , one edge joining e_1 and e_2 , and two loops at e_1 and e_2 respectively.

For polyhedra, a **flag** is an incident triple of face, edge, and vertex. Such definition needs to be modified for polygonal cell complexes. Take Figure 1 as an example again. It has only one vertex, one edge, and one face, but we would like it to have six flags just as a usual triangle. In a polygon, each flag corresponds to a triangle in its barycentric subdivision. We can use this as an alternative definition of a flag, and this definition works for polygonal cell complexes as well. As Figure 1 shows, the shaded area is a flag of the dunce hat, and a dunce hat has six flags.

Highly symmetric polygonal complexes have been studied in [1, 2, 9, 10]. In particular, simply-connected flag-transitive polygonal complexes with complete graphs as links are classified in [2]. The main motivation of this paper is to use these flag-transitive complexes to generate more flag-transitive complexes. More specifically, we would like to develop a product of complexes which preserves flag-transitivity, and the link of the product is some graph product of the links of factors.

2 Graph Tensor Product

Suppose that \bullet is certain type of graph product such that $V(\Gamma \bullet \Gamma') = V(\Gamma) \times V(\Gamma')$, and we want to define a complex product $*$ with the following property: for any complexes X and X' , and for any vertices $v \in X$ and $v' \in X'$, we have

$$L(X, v) \bullet L(X', v') \cong L(X * X', (v, v')).$$

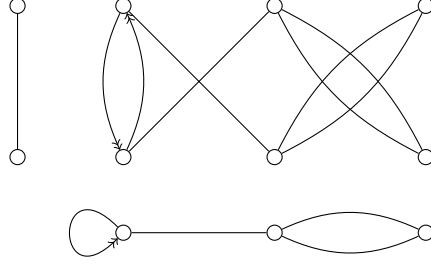


Figure 2: tensor product of non-simple graphs

Here we have already assumed that $V(X * X') = V(X) \times V(X')$. The above property provides sufficient information about how the complex product $*$ shall be defined. If we assume the 1-skeletons of X and X' are simple graphs, by considering the vertex sets of two link graphs in the equation, we have

$$\{\text{neighbours of } v \text{ in } X\} \times \{\text{neighbours of } v' \text{ in } X'\} = \{\text{neighbours of } (v, v') \text{ in } X * X'\},$$

which can be interpreted as two vertices (v, v') and (u, u') are adjacent in $X * X'$ if and only if v is adjacent to u in X and v' is adjacent to u' in X' . This is essentially the definition of the direct product of simple graphs. Since the 1-skeletons of complexes are not necessarily simple, we shall generalize the direct product to suit arbitrary graphs.

Definition 2.1. Suppose that Γ and Γ' are two arbitrary graphs with edge sets $E(\Gamma) = \{e_\alpha \mid \alpha \in A\}$ and $E(\Gamma') = \{e_\beta \mid \beta \in B\}$. The **tensor product** of Γ and Γ' , denoted by $\Gamma \otimes \Gamma'$, is a graph with vertex set $V(\Gamma \otimes \Gamma') = V(\Gamma) \times V(\Gamma')$, and edge set

$$E(\Gamma \otimes \Gamma') = \{e_{\alpha, \beta}^\delta \mid \alpha \in A, \beta \in B, \delta \in \{0, 1\}\},$$

where $e_{\alpha, \beta}^\delta$ is an edge joining (v_0, v'_δ) and $(v_1, v'_{1-\delta})$, given e_α joins v_0 and v_1 in Γ , and e_β joins v'_0 and v'_1 in Γ' .

Note that for simple graphs, the tensor product defined above is exactly the direct product of graphs. Like direct product, each pair of edges from two factors generates two edges in the tensor product, even when loops are involved, as illustrated in Figures 2 and 3. In some literatures such as [4], direct product is defined over graphs without parallel edges but admitting loops. In such definition, a loop serves as the identity of direct product. In particular a loop times an edge is an edge, and a loop times a loop is again a loop, while in our definition a loop times an edge is two parallel edges, and a loop times a loop creates two loops around the same vertex. Since we will need such direct product later, we take a different name and symbol for our generalized product.

There are some reasons to define tensor product in this manner. First, note the number of vertices in $L(X, v)$ is exactly the valency of v in X , where a loop at v contributes 2 to the number. Assuming $L(X, v) \bullet L(X', v') \cong L(X * X', (v, v'))$, this implies

$$d_X(v) \cdot d_{X'}(v') = d_{X * X'}((v, v')),$$

which is true for the tensor product, but not for the direct product admitting loops. Secondly, when we glue a face along a loop, the orientation of gluing matters, and the

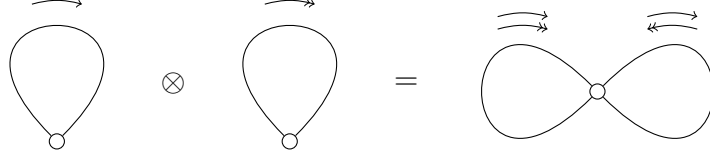


Figure 3: tensor product of two loops

tensor product can keep track of such orientations. In Definition 2.1, when e_α or e_β is a loop, we shall think of it as an edge joining two different ends of the loop, say $+$ and $-$, and label two ends of $e_{\alpha,\beta}^\delta$ by $+$ and $-$ accordingly. We can then lift any given orientation of a loop in a factor to edges generated by this loop in the product, as illustrated in Figures 2 and 3. This also allows us to define projections unambiguously. Note that we do not assume graphs to be directed. We just distinguish two ends of each loop.

Definition 2.2. Assume the notation of Definition 2.1. The **projection** from $\Gamma \otimes \Gamma'$ to Γ , denoted by π_Γ , is a continuous function such that π_Γ maps $(v, v') \in V(\Gamma \otimes \Gamma')$ to $v \in V(\Gamma)$, and $e_{\alpha,\beta}^\delta \in E(\Gamma \otimes \Gamma')$ to $e_\alpha \in E(\Gamma)$ isometrically between endpoints. The projection $\pi_{\Gamma'}$ from $\Gamma \otimes \Gamma'$ to Γ' is likewise defined.

The projections defined above are graph homomorphisms in the following sense.

Definition 2.3. Let Γ and Γ' be two arbitrary graphs. A continuous function φ from Γ to Γ' is a **homomorphism** if φ maps each vertex of Γ to a vertex of Γ' , and each open edge of Γ isometrically onto an open edge of Γ' .

Remark. In the above definition, the continuity of φ is essentially saying that a homomorphism maps incident vertices and edges to incident vertices and edges. Meanwhile, the isometric condition helps to choose a representative from all homotopic maps.

Note that the composition of two graph homomorphisms is again a graph homomorphism. Together with the trivial automorphisms, the class of graphs forms a category. The following proposition shows that the tensor product defined above is actually the categorical product of this category.

Proposition 2.4. Let Γ and Γ' be two arbitrary graphs. Suppose that Γ_0 is a graph with two homomorphisms $\varphi : \Gamma_0 \rightarrow \Gamma$ and $\varphi' : \Gamma_0 \rightarrow \Gamma'$. Then there exists a unique homomorphism $\psi : \Gamma_0 \rightarrow \Gamma \otimes \Gamma'$ such that $\varphi = \pi_\Gamma \circ \psi$ and $\varphi' = \pi_{\Gamma'} \circ \psi$. In other words, there exists a unique ψ such that the diagram in Figure 4 commutes.

Proof. Assume that there exists a continuous function $\psi : \Gamma_0 \rightarrow \Gamma \otimes \Gamma'$ such that $\varphi = \pi_\Gamma \circ \psi$ and $\varphi' = \pi_{\Gamma'} \circ \psi$. Then $\forall v \in V(\Gamma_0)$, we have $\varphi(v) = \pi_\Gamma \circ \psi(v)$ and $\varphi'(v) = \pi_{\Gamma'} \circ \psi(v)$. By Definition 2.2, we know that $\psi(v) = (\varphi(v), \varphi'(v))$.

Suppose that e is an open edge joining v and u in Γ_0 , and we denote $\varphi(e)$ and $\varphi'(e)$ by e_α and e_β respectively. By the continuity of ψ , $\psi(e)$ is an open path connecting $(\varphi(v), \varphi'(v))$ and $(\varphi(u), \varphi'(u))$. Notice that $e_\alpha = \varphi(e) = \pi_\Gamma \circ \psi(e)$ and $e_\beta = \varphi'(e) = \pi_{\Gamma'} \circ \psi(e)$. By Definition 2.2, we know $\psi(e)$ is either $e_{\alpha,\beta}^0$ or $e_{\alpha,\beta}^1$, determined by endpoints $(\varphi(v), \varphi'(v))$ and $(\varphi(u), \varphi'(u))$. In case e_α or e_β is a loop, by keeping track of ends of the loop, $\psi(e)$ is also uniquely determined. Moreover, the local isometry over open edges of φ and π_Γ forces ψ to map e isometrically to $\psi(e)$. Note that we have explicitly constructed a continuous

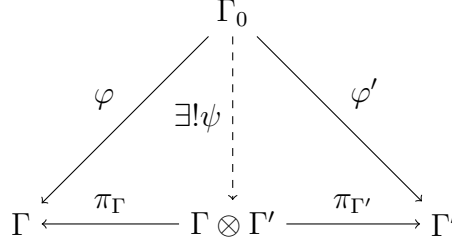


Figure 4: universal property of graph tensor product

ψ satisfying our initial assumption. We have also shown that ψ is uniquely determined, and actually a homomorphism, which finishes the proof. \square

For any two graphs Γ and Γ' , we denote the set of all homomorphisms from Γ to Γ' by $\text{Hom}(\Gamma, \Gamma')$. We have the following corollary about the number of homomorphisms.

Corollary 2.5. For any graphs $\Gamma, \Gamma_1, \Gamma_2$, we have

$$|\text{Hom}(\Gamma, \Gamma_1 \otimes \Gamma_2)| = |\text{Hom}(\Gamma, \Gamma_1)| \cdot |\text{Hom}(\Gamma, \Gamma_2)|.$$

Proof. An immediate consequence of Proposition 2.4. \square

Note that for any graph Γ , there is a homomorphism from Γ to a loop. Since we distinguish the orientations when we map an edge to a loop, there are actually 2^n such homomorphisms, where n is the number of edges of Γ . In particular, a loop is not the terminal object in the category of arbitrary graphs.

Corollary 2.6. Let Γ and Γ' be two graphs, P be a path in Γ of length n from v to u , and P' be a path in Γ' of length n from v' to u' . Then in $\Gamma \otimes \Gamma'$, there exists a unique path, denoted by $(P, P')_{\otimes}$, from (v, v') to (u, u') such that $\pi_{\Gamma}((P, P')_{\otimes}) = P$ and $\pi_{\Gamma'}((P, P')_{\otimes}) = P'$.

Proof. Let I be a graph which is a path of length n . We can give I a specific orientation from one end to the other. Then there is a natural homomorphism φ from I to P , as well as one φ' from I to P' . By Proposition 2.4, there exists a unique homomorphism $\psi : I \rightarrow \Gamma \otimes \Gamma'$ such that $\varphi = \pi_{\Gamma} \circ \psi$ and $\varphi' = \pi_{\Gamma'} \circ \psi$. Hence we have $P = \varphi(I) = \pi_{\Gamma} \circ \psi(I)$ and $P' = \varphi'(I) = \pi_{\Gamma'} \circ \psi(I)$. Note that $\psi(I)$ satisfies the conditions of $(P, P')_{\otimes}$, and the uniqueness of $(P, P')_{\otimes}$ follows the uniqueness of ψ . \square

Remark. For simple graphs, this result is straightforward from the definition of tensor product. This corollary clarifies the case when P or P' contains a loop, where the orientation going through the loop will determine the edge to choose in $(P, P')_{\otimes}$.

3 Complex Tensor Product

To define our complex product more concisely, we would like to extend the notation $(\ , \)_{\otimes}$ above. Let Γ_1 and Γ_2 be two graphs, C_1 be a cycle of length n in Γ_1 , and C_2 be a cycle of length m in Γ_2 . Both C_1 and C_2 are assigned initial vertices and orientations.

Specifically, C_2 is $(v_0, e_0, v_1, e_1, \dots, e_{m-1}, v_m = v_0)$, where $v_i \in V(\Gamma_2)$ and $e_j \in E(\Gamma_2)$. Then for $i \in \{0, 1, \dots, m-1\}$ we define

$$(C_1, C_2)_{\otimes}^{i\delta} := \left(\frac{[n, m]}{n} C_1, \frac{[n, m]}{m} C_2^{i\delta} \right)_{\otimes},$$

a cycle of length $[n, m]$ in $\Gamma_1 \otimes \Gamma_2$, where $[n, m]$ is the least common multiple of n and m , kC_j is the cycle repeating C_j k times, and C_2^{i0} is the same cycle as C_2 , but starting at v_i , while C_2^{i1} is the reversed cycle of C_2 starting at v_i .

Definition 3.1. Let X and Y be two polygonal cell complexes with face sets $F(X) = \{f_\alpha \mid \alpha \in A\}$ and $F(Y) = \{f_\beta \mid \alpha \in B\}$. We denote the boundary length of f_α and f_β by n_α and n_β respectively, and let (n_α, n_β) denote the greatest common divisor of n_α and n_β . The **tensor product** of X and Y , denoted by $X \otimes Y$, is a polygonal cell complex with 1-skeleton $X^1 \otimes Y^1$, the tensor product of the 1-skeletons of X and Y , and face set

$$F(X \otimes Y) = \{f_{\alpha, \beta}^{i\delta} \mid \alpha \in A, \beta \in B, i \in \{0, 1, \dots, (n_\alpha, n_\beta) - 1\}, \delta \in \{0, 1\}\},$$

where $f_{\alpha, \beta}^{i\delta}$ is a face attached along $(C_\alpha, C_\beta)_{\otimes}^{i\delta}$, while C_α is the cycle along which f_α is attached in X , and C_β is the cycle along which f_β is attached in Y .

Remark. We will use the jargon that $f_{\alpha, \beta}^{i\delta}$ is generated by f_α and f_β , especially when faces are not clearly indexed. In the above definition, note that $(C_\alpha, C_\beta)_{\otimes}^{i\delta}$ and $(C_\alpha, C_\beta)_{\otimes}^{i+(n_\alpha, n_\beta)\delta}$ are identical cycles with different starting vertices. To let a pair of corners of f_α and f_β contribute to exactly one face corner in $X \otimes Y$, we only choose $i \in \{0, 1, \dots, (n_\alpha, n_\beta) - 1\}$. Here we discard repeated corner pairs, not faces in $X \otimes Y$ attached along the same cycle. For example, let X and Y be 15-gons wrapped around a cycle of length 3 and 5 respectively. Note that the tensor product of a triangle and a pentagon is not the same as $X \otimes Y$. The former has only $2 \cdot (3, 5) = 2$ faces, while $X \otimes Y$ has $2 \cdot (15, 15) = 30$ faces in two groups, each of which has 15 faces with cyclically identical attaching maps.

In the example of a triangle tensor a pentagon, the only two faces meet at every vertex in the product. In general, when $n_\alpha \neq n_\beta$, two faces $f_{\alpha, \beta}^{i0}$ and $f_{\alpha, \beta}^{i1}$ meet at more than one vertex. Therefore the tensor product of two polygonal complexes is not necessarily polygonal. How about the case when $n_\alpha = n_\beta$? For n_α even, note that $f_{\alpha, \beta}^{i0}$ and $f_{\alpha, \beta}^{i1}$ have two vertices $(0, 0)$ and $(\frac{n_\alpha}{2}, \frac{n_\alpha}{2})$ in common, and the tensor product is not polygonal. For odd cases, we have the following result.

Proposition 3.2. Suppose that X and Y are polygonal complexes with all faces of the same odd length n . Then the tensor product $X \otimes Y$ is a polygonal complex.

Proof. Since X and Y are polygonal complexes, we know that X^1 and Y^1 are simple graphs, and hence the 1-skeleton of $X \otimes Y$, namely $X^1 \otimes Y^1$, is a simple graph as well. Consider the boundary of an arbitrary face $f_{\alpha, \beta}^{i\delta}$ in $X \otimes Y$, namely $(C_\alpha, C_\beta)_{\otimes}^{i\delta}$. Note that C_α and C_β are both simple closed cycles of the same length n , as they are boundaries of faces of polygonal complexes. Therefore $(C_\alpha, C_\beta)_{\otimes}^{i\delta}$ is a simple closed cycle of length n . In brief, every face of $X \otimes Y$ is attached along a simple closed cycle.

Now all we have to show is that the intersection of two faces in $X \otimes Y$ is either empty, a vertex, or an edge in $X \otimes Y$. Suppose that there exist two faces $f_{\alpha, \beta}^{i\delta}$ and $f_{\alpha', \beta'}^{j\delta'}$ in $X \otimes Y$

such that the intersection of $f_{\alpha,\beta}^{i^\delta}$ and $f_{\alpha',\beta'}^{j^{\delta'}}$ is neither empty, a vertex, nor an edge. For the case of $n = 3$, it is not hard to see that $f_{\alpha,\beta}^{i^\delta}$ and $f_{\alpha',\beta'}^{j^{\delta'}}$ share the same boundary, and in fact are the same face by the polygonality of X and Y . For the case of odd $n > 3$, note that $f_{\alpha,\beta}^{i^\delta}$ and $f_{\alpha',\beta'}^{j^{\delta'}}$ share two vertices which are not consecutive on the boundary of faces. By the polygonality of X and Y , this implies that $f_\alpha = f_{\alpha'}$ and $f_\beta = f_{\beta'}$. Consider the boundaries of $f_{\alpha,\beta}^{i^\delta}$ and $f_{\alpha,\beta}^{j^{\delta'}}$, namely $(C_\alpha, C_\beta)_\otimes^{i^\delta}$ and $(C_\alpha, C_\beta)_\otimes^{j^{\delta'}}$. When $\delta = \delta'$ and $i \neq j$, $(C_\alpha, C_\beta)_\otimes^{i^\delta}$ and $(C_\alpha, C_\beta)_\otimes^{j^{\delta'}}$ have no vertex in common. When $\delta \neq \delta'$, notice that a common vertex of $(C_\alpha, C_\beta)_\otimes^{i^\delta}$ and $(C_\alpha, C_\beta)_\otimes^{j^{\delta'}}$ corresponds to an integer m such that

$$j + m \equiv i - m \pmod{n} \Leftrightarrow 2m = i - j \pmod{n},$$

which has a unique solution when n is odd. In other words, when $\delta \neq \delta'$, $(C_\alpha, C_\beta)_\otimes^{i^\delta}$ and $(C_\alpha, C_\beta)_\otimes^{j^{\delta'}}$ intersect at exactly one vertex. Since $f_{\alpha,\beta}^{i^\delta}$ and $f_{\alpha,\beta}^{j^{\delta'}}$ share two vertices, we can conclude that $\delta = \delta'$ and $i = j$. This finishes the proof. \square

The complex tensor product does not preserve simple connectedness either.

Proposition 3.3. Let X and Y be an n -gon and m -gon respectively, where n and m are two positive integers. Then $X \otimes Y$ is simply-connected if and only if $n = m = 1$.

Proof. When $n = m = 1$, the 1-skeleton of $X \otimes Y$ is a vertex with two loops, as illustrated in Figure 3, and $X \otimes Y$ has two faces attached along these two loops respectively. In this case, $X \otimes Y$ is actually contractible, and of course simply-connected.

Now suppose that n and m are not both equal to 1. Without loss of generality, we can assume $n \geq 2$. Note that X has n vertices, n edges, and 1 face, whereas Y has m vertices, m edges, and 1 face. By Definition 3.1, the complex $X \otimes Y$ has nm vertices, $2nm$ edges, and $2(n, m)$ faces. Therefore $X \otimes Y$ has Euler characteristic

$$\chi(X \otimes Y) = nm - 2nm + 2(n, m) = -nm + 2(n, m) \leq -2m + 2m = 0.$$

By the following lemma, we know $X \otimes Y$ is not simply-connected. \square

Lemma 3.4. Suppose X is a finite simply-connected polygonal cell complex. Then the Euler characteristic of X is at least 1.

Proof. Suppose X has v vertices, e edges, and f faces. First we find an arbitrary spanning tree T for the 1 skeleton of X , and then contract T to get a new complex X' , which is also simply-connected. Note that T has $v - 1$ edges, and therefore X' has 1 vertex, $e - v + 1$ edges, and f faces. The fundamental group $\pi_1(X')$, a trivial group, can be presented as a group with $e - v + 1$ generators and f relators. Consider the abelianization of $\pi_1(X')$, which is again trivial. Then the presentation can be expressed as f homogeneous equations of $e - v + 1$ unknowns over \mathbb{Z} . To have only trivial solution, the number of equations needs to be at least the number of unknowns. So we have $f \geq e - v + 1$, and therefore $v - e + f \geq 1$. \square

Remark. Let X and Y be two arbitrary complexes, and C be a cycle along the 1-skeleton of $X \otimes Y$. This proposition shows that the contractibility of $\pi_X(C)$ and $\pi_Y(C)$ does not guarantee the contractibility of C . Conversely, when C is contractible in $X \otimes Y$, can we

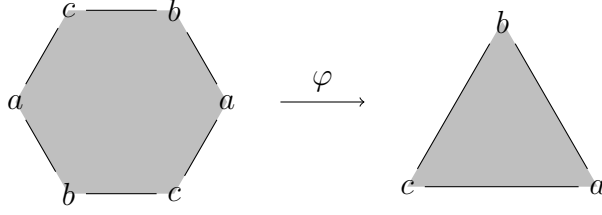


Figure 5: a complex homomorphism from a hexagon to a triangle

conclude that $\pi_X(C)$ and $\pi_Y(C)$ are contractible? The answer is positive. We can find a series $\{C_j\}$ of homotopic cycles of C such that $C_0 = C$, C_n is a vertex, and each C_j morphs through a single face $f_{\alpha,\beta}^{i_\delta}$ to obtain C_{j+1} . Note that $\pi_X(C_j)$ can morph through a single face f_α to obtain $\pi_X(C_{j+1})$, even when the length of f_α properly divides the length of $f_{\alpha,\beta}^{i_\delta}$. Therefore $\pi_X(C) = \pi_X(C_0)$ is homotopic to $\pi_X(C_n)$, which is a vertex.

In the above remark, we actually abuse the notation π_X , as we have not yet defined projection maps for complex tensor products. To define such projection maps, first we introduce some terminology. Let X and Y be an n -gon and m -gon with centre O_X and O_Y respectively. A function $\rho : X \rightarrow Y$ is **radial** if ρ sends O_X to O_Y , ∂X to ∂Y , and for every point $P \in \partial X$, every real number $t \in [0, 1]$, we have

$$\rho(t \cdot O_X + (1 - t)P) = t \cdot O_Y + (1 - t)\rho(P).$$

Definition 3.5. Assume the notation of Definition 3.1. The **projection** from $X \otimes Y$ to X , denoted by π_X , is a continuous function such that π_X restricted to $X^1 \otimes Y^1$ is exactly π_{X^1} , the projection of the graph tensor product, and π_X maps $f_{\alpha,\beta}^{i_\delta} \in F(X \otimes Y)$ radially to $f_\alpha \in F(X)$. The projection π_Y from $X \otimes Y$ to Y is likewise defined.

The projection maps defined above are complex homomorphisms in the following sense.

Definition 3.6. Let X and Y be two polygonal cell complexes. A continuous function φ from X to Y is a **homomorphism** if φ restricted to X^1 is a graph homomorphism to Y^1 , and φ maps each face of X radially to a face of Y and each open face corner (ignoring the boundary) of X homeomorphically to an open face corner of Y .

Remark. In the above definition, the continuity of φ is essentially saying that a complex homomorphism maps incident cells to incident cells. Similar to the isometric condition in graph homomorphism, the radial condition is imposed to rule out homotopic complex homomorphisms. Most important of all, the homeomorphic corner condition forces a face of X to wrap around a face f of Y along the direction of the attaching map of f , possibly more than once. In particular, a face of length n can only be mapped to a face of length dividing n . Figure 5 illustrates such phenomenon, where corners are mapped to a corner with the same label. The projection π_X of complex tensor product mapping $f_{\alpha,\beta}^{i_\delta}$ to f_α is also a typical example.

Note that the composition of two complex homomorphisms is again a complex homomorphism. Together with the trivial automorphisms, the class of polygonal cell complexes forms a category. The following proposition shows that the complex tensor product defined above is actually the categorical product of this category.

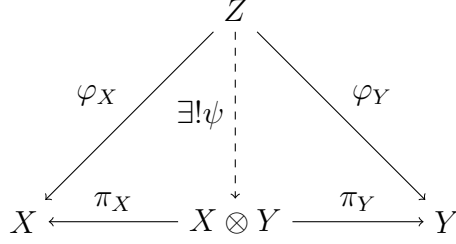


Figure 6: universal property of complex tensor product

Proposition 3.7. Let X and Y be two polygonal cell complexes. Suppose that Z is a complex with two homomorphisms $\varphi_X : Z \rightarrow X$ and $\varphi_Y : Z \rightarrow Y$. Then there exists a unique homomorphism $\psi : Z \rightarrow X \otimes Y$ such that $\varphi_X = \pi_X \circ \psi$ and $\varphi_Y = \pi_Y \circ \psi$. In other words, there exists a unique ψ such that the diagram in Figure 6 commutes.

Proof. Assume that there exists a continuous function $\psi : Z \rightarrow X \otimes Y$ such that $\varphi_X = \pi_X \circ \psi$ and $\varphi_Y = \pi_Y \circ \psi$. Note that φ_X , φ_Y , π_X , and π_Y restricted to the 1-skeletons of their domains are all graph homomorphisms. By Proposition 2.4, the restriction of ψ to Z^1 is a uniquely determined graph homomorphism to $X^1 \otimes Y^1$.

Suppose that f is a face in Z , $\varphi_X(f)$ wraps around a face f_α in X , and $\varphi_Y(f)$ wraps around a face f_β in Y . Then $\varphi_X(f) = \pi_X \circ \psi(f)$ wraps around f_α , and $\varphi_Y(f) = \pi_Y \circ \psi(f)$ wraps around f_β . By Definition 3.5, $\psi(f)$ must wrap around $f_{\alpha,\beta}^{i\delta}$ for some i and δ . Let c be a corner of f . Then we must have $\varphi_X(c) = \pi_X \circ \psi(c)$ and $\varphi_Y(c) = \pi_Y \circ \psi(c)$. By the remark after Definition 3.1, this pair of corners $(\varphi_X(c), \varphi_Y(c))$, orientation included, appears in exactly one $f_{\alpha,\beta}^{i\delta}$. Therefore i and δ are uniquely determined, and $\psi(f)$ wraps around this $f_{\alpha,\beta}^{i\delta}$. Moreover, the radially of φ_X and π_X forces ψ to map f radially to $f_{\alpha,\beta}^{i\delta}$. Note that we have explicitly constructed a continuous ψ satisfying our initial assumption. We have also shown that ψ is uniquely determined, and actually a complex homomorphism, which finishes the proof. \square

Remark. For any two complexes X and Y , we denote the set of all complex homomorphisms from X to Y by $\text{Hom}(X, Y)$. Similarly to Corollary 2.5, we have

$$|\text{Hom}(Z, X \otimes Y)| = |\text{Hom}(Z, X)| \cdot |\text{Hom}(Z, Y)|.$$

As we mentioned earlier, for any graph Γ , there is a homomorphism from Γ to a loop. It is reasonable to ask the following question: for any complex X , is there always a homomorphism from X to a 1-gon? The answer is negative. Take Figure 7 as an example. Once the image of the leftmost edge is determined, it determines the image of all other edges. If we identify the leftmost and the rightmost edges with a twist, i.e. making it a Mobius strip, then there is no way to have a homomorphism. Note that this question is not related to orientability. If the complex is a strip with 3 squares, then the Mobius case has a homomorphism, while the orientable case does not.

Proposition 3.8. Let X and Y be two polygonal cell complexes, and $\varphi : X \rightarrow Y$ be a complex homomorphism mapping a vertex $v \in V(X)$ to $u \in V(Y)$. Then φ induces a graph homomorphism $L(\varphi)$ from $L(X, v)$ to $L(Y, u)$. Moreover, let Z be another complex and $\rho : Y \rightarrow Z$ be a complex homomorphism mapping u to $w \in V(Z)$. Then we have $L(\rho \circ \varphi) = L(\rho) \circ L(\varphi)$, as illustrated in Figure 8.

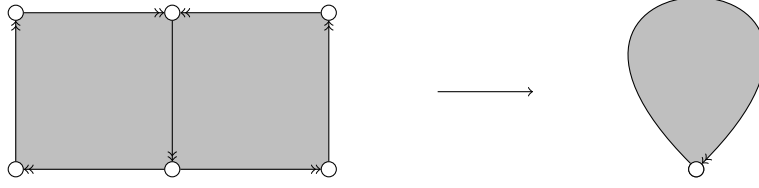


Figure 7: a homomorphism to a 1-gon

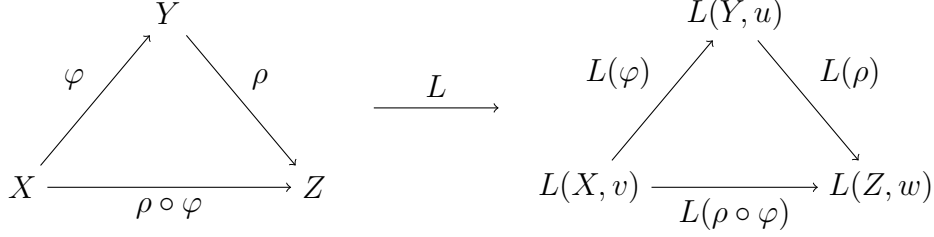


Figure 8: functoriality of L

Proof. By definition, $L(X, v)$ has vertices corresponding to edge ends around v in X , and edges corresponding to face corners at v in X . Since φ restricted to X^1 is a graph homomorphism, φ maps an edge end around v in X to an edge end around u in Y . In addition, by the homeomorphic condition in Definition 3.6, φ maps a face corner at v joining two edge ends around v homeomorphically to a face corner at u joining two edge ends around u . Therefore φ induces a graph homomorphism $L(\varphi)$ from $L(X, v)$ to $L(Y, u)$. Once these induced graph homomorphisms between link graphs are defined, the equality $L(\rho \circ \varphi) = L(\rho) \circ L(\varphi)$ follows immediately. \square

Remark. To each polygonal cell complex, we can assign a distinguished vertex to be the basepoint. Together with basepoint-preserving homomorphisms, the class of pointed polygonal cell complexes also forms a category. The above proposition is essentially saying that L is a functor from this category to the category of graphs.

Now we move back to the main purpose of this chapter: to develop a complex product interacting nicely with some product of link graphs. From the above discussion, we know that the complex tensor product arises naturally in the category of polygonal cell complexes. Does this natural categorical product fulfill the main job? Yes, it does.

Theorem 3.9. Suppose that X and Y are two polygonal cell complexes, and v and u are two vertices in X and Y respectively. Then we have

$$L(X, v) \otimes L(Y, u) \cong L(X \otimes Y, (v, u)).$$

Proof. We can identify edge ends incident to a vertex as paths of length 1 leaving the vertex, since a loop contributes to two edge ends as well as two such paths, which we call 1-paths for short. By Corollary 2.6, there is a bijection between 1-paths leaving (v, u) in $X \otimes Y$ and pairs of 1-path leaving v in X and 1-path leaving u in Y . Therefore we can index 1-paths leaving (v, u) in $X \otimes Y$ by such 1-path pairs in X and Y .

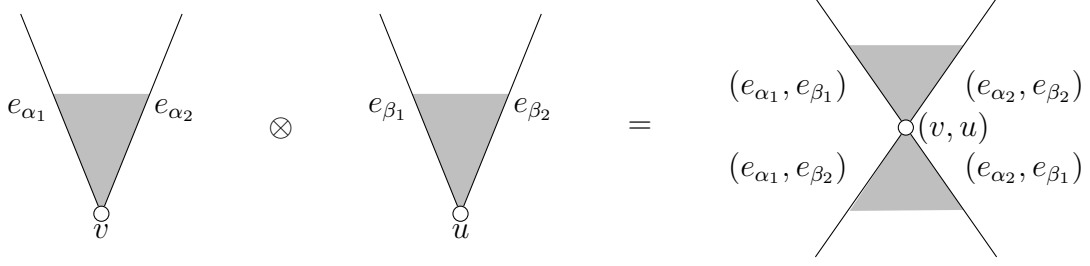


Figure 9: well linked tensor product

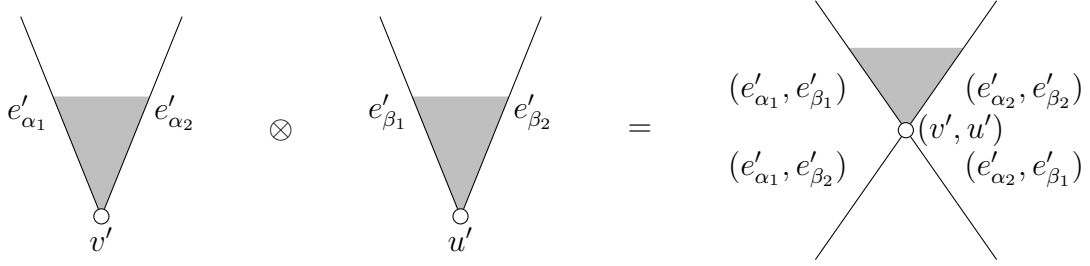


Figure 10: automorphic image of Figure 9

Suppose that $f_\alpha \in F(X)$ has a corner c_α at $(e_{\alpha_1}, v, e_{\alpha_2})$, and $f_\beta \in F(Y)$ has a corner c_β at $(e_{\beta_1}, u, e_{\beta_2})$, as illustrated in Figure 9. These e_* 's should be understood as 1-paths. By the remark after Definition 3.1, the pairing of these two corners appears exactly once in $f_{\alpha, \beta}^{i^0}$ and $f_{\alpha, \beta}^{j^1}$ respectively, forming corners $((e_{\alpha_1}, e_{\beta_1}), (v, u), (e_{\alpha_2}, e_{\beta_2}))$ and $((e_{\alpha_1}, e_{\beta_2}), (v, u), (e_{\alpha_2}, e_{\beta_1}))$ in $X \otimes Y$. Note that by taking projection maps, we know that any face corner at (v, u) comes from some pairing of corners at v and u .

Now we translate the above statements in terms of corresponding link graphs. First of all, we have $V(L(X, v)) \times V(L(Y, u)) \cong V(L(X \otimes Y, (v, u)))$. Secondly, the corner c_α is an edge joining vertices e_{α_1} and e_{α_2} in $L(X, v)$, and c_β is an edge joining vertices e_{β_1} and e_{β_2} in $L(Y, u)$. Notice that the edge pair (c_α, c_β) contributes to one edge joining $(e_{\alpha_1}, e_{\beta_1})$ and $(e_{\alpha_2}, e_{\beta_2})$, and one edge joining $(e_{\alpha_1}, e_{\beta_2})$ and $(e_{\alpha_2}, e_{\beta_1})$ in $L(X \otimes Y, (v, u))$. Meanwhile, taking all possible pairings of edges exhausts all edges in $L(X \otimes Y, (v, u))$. By Definition 2.1, this is exactly saying that $L(X, v) \otimes L(Y, u) \cong L(X \otimes Y, (v, u))$. \square

Remark. In the terminology of category theory, this theorem is essentially saying that the functor L from the category of pointed complexes to the category of graphs preserves categorical products, which is not always true for an arbitrary functor.

As indicated in Propositions 3.2 and 3.3, the complex tensor product does not necessarily preserve polygonality and simple connectedness. Fortunately, complex tensor product does preserve the most important property for our purpose.

Theorem 3.10. Let X and Y be any two flag-transitive polygonal cell complexes. Then the complex tensor product $X \otimes Y$ is flag-transitive.

Proof. In case X or Y has no faces, then $X \otimes Y$ is simply a graph, and the flag-transitivity follows easily from the definition of graph tensor product. Hereafter we assume that both X and Y have at least one face.

Let $((e_{\alpha_1}, e_{\beta_1}), (v, u), (e_{\alpha_2}, e_{\beta_2}))$ be a face corner in $X \otimes Y$, which projects to a corner $(e_{\alpha_1}, v, e_{\alpha_2})$ in X and a corner $(e_{\beta_1}, u, e_{\beta_2})$ in Y , as illustrated in Figure 9. Let $((e'_{\alpha_1}, e'_{\beta_1}), (v', u'), (e'_{\alpha_2}, e'_{\beta_2}))$ be another face corner in $X \otimes Y$, which projects to a corner $(e'_{\alpha_1}, v', e'_{\alpha_2})$ in X and a corner $(e'_{\beta_1}, u', e'_{\beta_2})$ in Y , as illustrated in Figure 10. Since X and Y are flag-transitive, there exist $\rho \in \text{Aut}(X)$ mapping $(e_{\alpha_1}, v, e_{\alpha_2})$ to $(e'_{\alpha_1}, v', e'_{\alpha_2})$ and $\sigma \in \text{Aut}(Y)$ mapping $(e_{\beta_1}, u, e_{\beta_2})$ to $(e'_{\beta_1}, u', e'_{\beta_2})$. Comparing Figures 9 and 10, note that (ρ, σ) gives an automorphism of $X \otimes Y$ mapping $((e_{\alpha_1}, e_{\beta_1}), (v, u), (e_{\alpha_2}, e_{\beta_2}))$ to $((e'_{\alpha_1}, e'_{\beta_1}), (v', u'), (e'_{\alpha_2}, e'_{\beta_2}))$. The above discussion shows that $\text{Aut}(X \otimes Y)$ acts transitively on face corners with orientations, and therefore transitively on half-corners. In other words, $\text{Aut}(X \otimes Y)$ acts transitively on flags. \square

Remark. In Figure 9, flipping both corners in X and Y will flip both corners in $X \otimes Y$, whereas flipping only one corner in either X or Y will swap two corners in $X \otimes Y$.

4 Factorization and Symmetry

In the proof of Theorem 3.10, the key fact we used is the following relation:

$$\text{Aut}(X) \times \text{Aut}(Y) \leq \text{Aut}(X \otimes Y).$$

Is it possible that these two groups are actually isomorphic? When X and Y are isomorphic, we can swap X and Y to obtain an extra automorphism, since the complex tensor product is commutative up to isomorphism. In addition to swapping, the following proposition gives more extra automorphisms in a less obvious way.

Proposition 4.1. Let X , Y , and Z be polygonal cell complexes. Then we have

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z).$$

In other words, complex tensor product is associative up to isomorphism.

Proof. A categorical result of the universal property in Proposition 3.7. See [6]. \square

The associativity of the complex tensor product complicates $\text{Aut}(X \otimes Y)$. For example, if Y can be factorized into $X \otimes Z$, then $X \otimes Y \cong X \otimes (X \otimes Z)$ has an automorphism swapping the two copies of X . Hence the symmetry of the product of complexes is also related to the factoring of complexes. In response to associativity, we modify the original question as follows: for complexes X_i which are irreducible with respect to complex tensor product, is the automorphism group $\text{Aut}(\otimes X_i)$ generated by automorphisms of X_i 's, together with permutations of isomorphic factors? By a **Cartesian automorphism**, we mean an element in the subgroup of $\text{Aut}(\otimes X_i)$ generated in the above manner.

There have been lots of studies about the symmetry of different products of graphs. One of the major goals of this chapter is to apply the theory of the graph direct product to the complex tensor product. Hence we first introduce related theorems about the graph direct product. The book [4] by Hammack, Imrich, and Klavžar offers a comprehensive survey of products of graphs, and we shall follow their approach and terminology here.

We briefly mentioned the direct product of graphs in Chapter 2. Here we give the definition again, with an emphasis on the possible presence of loops. We say that a graph Γ is a **simple graph with loops admitted** if for any $u, v \in V(\Gamma)$, there is at most one

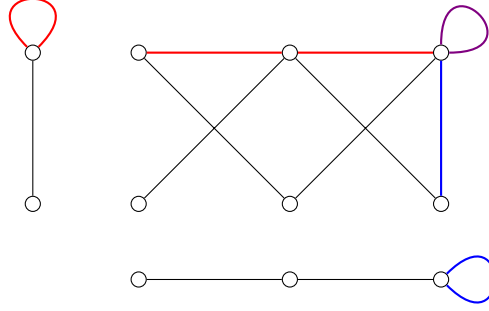


Figure 11: direct product of graphs in \mathfrak{S}_0

edge joining u and v , including the case $u = v$. In particular, there is at most one loop at a vertex. For convenience, we use \mathfrak{S} to denote the class of simple graphs, and \mathfrak{S}_0 to denote the class of simple graphs with loops admitted.

Definition 4.2. Let Γ and Γ' be two graphs in \mathfrak{S}_0 . The **direct product** of Γ and Γ' , denoted by $\Gamma \times \Gamma'$, is a graph in \mathfrak{S}_0 with vertex set $V(\Gamma \times \Gamma') = V(\Gamma) \times V(\Gamma')$. There is an edge joining two vertices (v, v') and (u, u') in $\Gamma \times \Gamma'$ if and only if there is an edge joining v and u in Γ , and there is an edge joining v' and u' in Γ' .

Note in the above definition, v and v' could be the same vertex, as well as u and u' . Figure 11 illustrates the direct product of two graphs in \mathfrak{S}_0 . Under this definition, notice that a loop L serves as the identity element of direct product of graphs. In other words, for any simple graph Γ with loops admitted, we always have

$$L \times \Gamma \cong \Gamma \times L \cong \Gamma.$$

Also note that the direct product of two edges is again two edges, laid out as a cross in the figure, which is part of the reason why graph theorists choose the symbol “ \times ” [4]. Therefore the direct product of two connected graphs is not necessarily connected. The following theorem is known as Weichsel’s Theorem [4].

Theorem 4.3. Suppose that Γ and Γ' are two connected simple graphs with at least two vertices. If Γ and Γ' are both bipartite, then $\Gamma \times \Gamma'$ has exactly two components. If at least one of Γ and Γ' is not bipartite, then $\Gamma \times \Gamma'$ is connected.

Proof. The first part of the theorem is straightforward. For the second part, note that a simple graph is not bipartite if and only if there is an odd cycle in the graph. By exploiting such a cycle properly, the second part of the theorem follows. For a detailed proof, please refer to Theorem 5.9 in [4]. \square

A graph Γ is **prime** if Γ has more than one vertex, and $\Gamma \cong \Gamma_1 \times \Gamma_2$ implies that either Γ_1 or Γ_2 is a loop. Note that the idea of being prime depends on the class of graphs we are talking about. For example, let Γ be a path of length 3, which has 4 vertices. Then Γ is prime in \mathfrak{S} , as the only possible factoring is the product of two edges, which is the disjoint union of two edges. And the statement that $\Gamma \cong \Gamma_1 \times \Gamma_2$ implies either Γ_1 or Γ_2 is a loop is still logically true. However, Γ can be factorized in \mathfrak{S}_0 as the graph on the left of Figure 11 times one edge in the bottom, and hence Γ is not prime in \mathfrak{S}_0 .

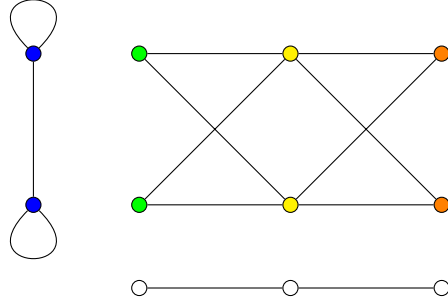


Figure 12: vertices with the same set of neighbours

Consider the question of factoring a graph into the product of prime graphs. For a finite graph, such a prime factorization always exists, since the number of vertices of factors decreases as the factoring goes. However, such a prime factorization is not necessarily unique, and it depends on the graph itself and the class of graphs where we do the factoring. For example, a path of length 3 together with associativity can be used to create graphs with non-unique prime factorizations in \mathfrak{S} . There are also graphs with non-unique prime factorizations in \mathfrak{S}_0 , an example of which can be found in [4]. The following theorem of unique prime factorization is due to McKenzie [8].

Theorem 4.4. Suppose that $\Gamma \in \mathfrak{S}_0$ is a finite connected non-bipartite graph with more than one vertex. Then Γ has a unique factorization into primes in \mathfrak{S}_0 .

The next question is about the automorphism group of direct product, which hopefully has only these Cartesian automorphisms with respect to the product. Note that a pair of vertices with the same set of neighbours creates pairs of vertices with the same set of neighbours in the direct product, and results in lots of non-Cartesian automorphisms. This phenomenon is illustrated in Figure 12, where a vertex with a loop should have itself as a neighbour. We say that a graph is *R-thin* if there are no vertices with the same set of neighbours. In addition to *R-thinness*, the disconnectedness due to Theorem 4.3 also creates non-Cartesian automorphisms. Even when the direct product is connected, there might still be some exotic automorphisms. The following theorem is due to Dörfler [3].

Theorem 4.5. Suppose that $\Gamma \in \mathfrak{S}_0$ is a finite connected non-bipartite *R-thin* graph with a prime factorization $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ in \mathfrak{S}_0 . Then $\text{Aut}(\Gamma)$ is generated by automorphisms of prime factors and permutations of isomorphic factors.

We would like to use Theorems 4.4 and 4.5 to develop similar results for the complex tensor product. The first problem we immediately encounter is that, for the complex tensor product, we obtain the 1-skeleton of the product through the graph tensor product, which is not exactly the same as the direct product of graphs. Fortunately, such a difference does not really take place in graphs with higher symmetries.

Proposition 4.6. Let $\Gamma \in \mathfrak{S}_0$ be a finite connected non-bipartite *R-thin* graph with more than one vertex, and $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ be the unique prime factorization in \mathfrak{S}_0 . If Γ is edge-transitive, then Γ and each prime factor Γ_i are in \mathfrak{S} .

Proof. Since Γ has more than one vertex, the connectedness of Γ implies that Γ has a non-loop edge. By the edge-transitivity of Γ , we know Γ has no loop, and hence is in \mathfrak{S} . If each factor Γ_i has a loop, then the product Γ will have a loop, which is not true. If each factor Γ_i is loop-free, then we have finished the proof. Hence we can assume there is at least one factor with a loop, and at least one factor without a loop.

Let Γ_α be the direct product of all factors with a loop, and Γ_β be the direct product of all factors without a loop. Then we have $\Gamma = \Gamma_\alpha \times \Gamma_\beta$. Note that permuting isomorphic factors of Γ does not involve permuting factors of Γ_α with factors of Γ_β . By Theorem 4.5, we have $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_\alpha) \times \text{Aut}(\Gamma_\beta)$. Since a prime factor has more than one vertex, Γ_α and Γ_β both have more than one vertex. Since Γ is connected, Γ_α and Γ_β are both connected. Hence Γ_α has a loop at some vertex v and a non-loop edge joining two vertices v_α and v'_α , while Γ_β has a non loop edge joining two vertices v_β and v'_β . Then in $\Gamma = \Gamma_\alpha \times \Gamma_\beta$, there is an edge joining (v, v_β) and (v, v'_β) , and another edge joining (v_α, v_β) and (v'_α, v'_β) . Notice that $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_\alpha) \times \text{Aut}(\Gamma_\beta)$ can not send the first edge to the second one, contradicting the assumption that Γ is edge-transitive. \square

Remark. To visually interpret the last few lines of the proof, it says that a Cartesian automorphism can not permute horizontal edges with slant edges in Figure 12.

Now we move on to the factorization of polygonal cell complexes. First consider the following example. Let X and Y be a triangle and a pentagon respectively, X' be a cycle of length 3 with two triangles attached, and Y' be a cycle of length 5 with two pentagons attached. Since the numbers of vertices of these complexes are prime, the only possible way to factorize them is to have a factor of one vertex with at least a loop and a face, which creates double edges in the product. Hence we know these complexes can not be factorized further, and we have non-unique factorizations $X \otimes Y' \cong X' \otimes Y$.

Here we give another example of non-unique factorization. Let X be a triangle, and Y' be a $(7 \cdot 5)$ -gon wrapped around a cycle of length 5. By Definition 3.1, since 3 and $7 \cdot 5$ are coprime, $X \otimes Y'$ has two faces of length $3 \cdot 5 \cdot 7$, wrapped around two cycles of length $3 \cdot 5$ for 7 rounds. Consider a $(7 \cdot 3)$ -gon X' wrapped around a cycle of length 3, and a pentagon Y . It is easy to see that $X \otimes Y' \cong X' \otimes Y$, and these complexes can not be factorized further. To avoid these non-uniquely factorized situations, we restrict our discussion to the factorization of simple complexes.

Definition 4.7. A polygonal cell complex X is a **simple** complex if X has at least one face, X has no pairs of faces attached along the same cycle, and the attaching map of each face does not wrap around a cycle more than once. A polygonal cell complex X is a **prime** complex if there do not exist complexes X_1 and X_2 such that $X = X_1 \otimes X_2$.

Remark. Figure 13 above is a simple complex with two 1-gons. If we add another 2-gon attached along two different loops, the resulting complex is still a simple complex, as the boundary cycles of these faces are not exactly the same.

To factorize a complex X , our general setting is as follows. We assume that we know a factorization of the 1-skeleton $X^1 = \Gamma_1 \otimes \Gamma_2$, and try to find a complex factorization $X = X_1 \otimes X_2$ such that $X_1^1 = \Gamma_1$ and $X_2^1 = \Gamma_2$. A natural thought is to project the faces of X down to Γ_1 and Γ_2 to be faces. Consider the complex tensor product of a triangle and a pentagon, which is a complex with two 15-gons. Note that when we project these two 15-gons back to the 1-skeletons of factors, what we obtain are 15-gons wrapped around cycles of length 3 and 5 respectively, not the original faces.

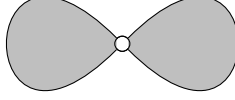


Figure 13: a simple complex with two 1-gons

Definition 4.8. Let X be a polygonal cell complex, f be a face of X attached along a cycle C_f , and Γ_1 and Γ_2 be two graphs such that $X^1 = \Gamma_1 \otimes \Gamma_2$. The **reductive projection** of f to Γ_i , denoted by $\pi_{\Gamma_i}(f)$, is a face attached along the reduced cycle of $\pi_{\Gamma_i}(C_f)$ in Γ_i , namely the shortest cycle C such that repeating C gives $\pi_{\Gamma_i}(C_f)$.

Remark. In exactly the same way, we can define $\pi_{\Gamma_i}(f)$ for the case $X^1 = \otimes_{i=1}^n \Gamma_i$. Note that when $X^1 = \Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3$, we have $\pi_{\Gamma_1}(f) = \pi_{\Gamma_1}(\pi_{\Gamma_1 \otimes \Gamma_3}(f)) = \pi_{\Gamma_1}(\pi_{\Gamma_1 \otimes \Gamma_2}(f))$.

Proposition 4.9. Let X be a simple complex, and Γ_1 and Γ_2 be two graphs such that $X^1 = \Gamma_1 \otimes \Gamma_2$. If there exist two complexes X_1 and X_2 with 1-skeletons Γ_1 and Γ_2 respectively such that $X = X_1 \otimes X_2$, then X_1 and X_2 are simple complexes whose faces are precisely the reductive projections of faces of X .

Proof. Suppose that such complexes X_1 and X_2 exist. Let f be a face of X attached along a cycle C_f of length n , and let C_j of length n_j be the reduced cycle of $\pi_{\Gamma_j}(C_f)$ in X_j for $j \in \{1, 2\}$. Note that f is generated by a face f_1 of X_1 attached along $m_1 C_1$, and by a face f_2 of X_2 attached along $m_2 C_2$, where $m_i C_i$ is the cycle made by repeating C_i for m_i times. By Definition 3.1, f_1 and f_2 generate faces attached along $(m_1 C_1, m_2 C_2)_{\otimes}^{i\delta}$, where $i \in \{0, 1, \dots, (m_1 n_1, m_2 n_2) - 1\}$ and $\delta \in \{0, 1\}$. By the Euclidean algorithm, we can find an integer $k > 0$ such that $k \equiv 0 \pmod{n_1}$ and $k \equiv (n_1, n_2) \pmod{n_2}$. Note that in k steps along $(m_1 C_1, m_2 C_2)_{\otimes}^{0\delta}$, we can walk from the starting vertex of $(m_1 C_1, m_2 C_2)_{\otimes}^{0\delta}$ to the starting vertex of $(m_1 C_1, m_2 C_2)_{\otimes}^{(n_1, n_2)\delta}$, so these two cycles are identical. Since X is simple, there are no pairs of faces attached along the same cycle in X . Therefore we have $(n_1, n_2) \geq (m_1 n_1, m_2 n_2) \geq (n_1, n_2)$. Now consider the length of the face f , which is

$$n = [m_1 n_1, m_2 n_2] = \frac{m_1 n_1 \cdot m_2 n_2}{(m_1 n_1, m_2 n_2)} = \frac{m_1 m_2 \cdot n_1 n_2}{(n_1, n_2)} = m_1 m_2 \cdot [n_1, n_2].$$

This shows that f is attached along some cycle $(C_1, C_2)_{\otimes}^{i\delta}$ of length $[n_1, n_2]$ for $m_1 m_2$ rounds, and the simplicity of X implies that $m_1 = m_2 = 1$. In other words, X_i must have the reductive projection $\pi_{\Gamma_i}(f)$ of f as its face. Note that different faces of X might have the same reductive projection in X_i , and we have to discard duplicated ones. Otherwise duplicated faces in X_i will generate duplicated faces in X , violating the simplicity of X . Conversely, any faces f_1 of X_1 and f_2 of X_2 are the reductive projections of the faces in X they generate. Hence X_1 and X_2 are the simple complexes with exactly those faces from the reductive projections of faces of X . \square

Proposition 4.10. Let X , X_1 , and X_2 be polygonal cell complexes such that $X = X_1 \otimes X_2$. Then X is a simple complex if and only if X_1 and X_2 are simple complexes.

Proof. Proposition 4.9 takes care of the only if part, and here we prove the if part. Suppose that X has an n -gon f attached along a cycle for m rounds. Since X_1 and X_2 are simple, f

must be generated by the reductive projections of f to X_1^1 and X_2^1 , which are of length l_1 and l_2 respectively. Note that l_1 and l_2 both divide $\frac{n}{m}$. Then the two reductive projections generate faces of length $n = [l_1, l_2] \leq \frac{n}{m}$. Hence we can conclude that $m = 1$. If there is another face f' in X attached along the same cycle with f , then f' is also generated by the reductive projections of f . If we can show a face in X_1 and a face in X_2 do not generate duplicated faces in X , then this implies X is a simple complex.

Suppose that a face f_1 of X_1 has vertices $v_0, v_1, \dots, v_{p-1}, v_0$ in order, and a face f_2 of X_2 has vertices $u_0, u_1, \dots, u_{q-1}, u_0$ in order. By the remark after Definition 3.1, every pair of corners of f_1 and f_2 appears exactly once in the faces generated by f_1 and f_2 . If two faces generated by f_1 and f_2 are attached along the same cycle in X , there must be two pairs of corners of f_1 and f_2 forming the same corner in X . In particular, we can find $(v_i, u_{i'}) = (v_j, u_{j'})$ such that $i \neq j$ or $i' \neq j'$. When $i \neq j$, we have $v_i = v_j$ and $v_{i+k} = v_{j+k}$ for any integer $k \bmod p$. This implies that f_1 wraps around a cycle more than once, violating the simplicity of X_1 . Similarly $i' \neq j'$ contradicts the simplicity of X_2 . The contradiction results from the assumption that two faces generated by f_1 and f_2 are attached along the same cycle in X . Hence we know that f_1 and f_2 does not generate duplicated faces, and the simplicity of X follows. \square

Proposition 4.11. Let X be a simple complex, and Γ_1 and Γ_2 be two graphs such that $X^1 = \Gamma_1 \otimes \Gamma_2$. Then the following two statements are equivalent:

- (1) There exist two complexes X_1 and X_2 such that $X_i^1 = \Gamma_i$ and $X = X_1 \otimes X_2$.
- (2) For any faces f_1 and f_2 of X , X contains all faces generated by $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$.

Proof. Assume (1). By Proposition 4.9, X_1 and X_2 are the simple complexes with exactly those reductive projections of X as faces. For any faces f_1 and f_2 of X , $\pi_{\Gamma_1}(f_1)$ is a face of X_1 , and $\pi_{\Gamma_2}(f_2)$ is a face of X_2 . Since $X = X_1 \otimes X_2$, X contains all faces generated by $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$. Hence (1) implies (2).

Assume (2). First we show that a face f of X can be generated by $\pi_{\Gamma_1}(f)$ and $\pi_{\Gamma_2}(f)$. Let C_f , C_1 , and C_2 be the boundary cycles of f , $\pi_{\Gamma_1}(f)$, and $\pi_{\Gamma_2}(f)$ respectively. By Definition 4.8, we can assume that $\pi_{\Gamma_j}(C_f) = n_j C_j$ for $j \in \{1, 2\}$, namely repeating C_j for n_j times gives $\pi_{\Gamma_j}(C_f)$. Note that f is attached along some cycle $(n_1 C_1, n_2 C_2)_{\otimes}^{i^\delta}$, which can be rewritten as $(n_1, n_2) \left(\frac{n_1}{(n_1, n_2)} C_1, \frac{n_2}{(n_1, n_2)} C_2 \right)_{\otimes}^{i^\delta}$. Since the simple complex X has no face attached around a cycle more than once, we know that $(n_1, n_2) = 1$, and therefore

$$\text{length } C_f = n_1 \cdot (\text{length } C_1) = n_2 \cdot (\text{length } C_2) = [\text{length } C_1, \text{length } C_2].$$

This shows that $\pi_{\Gamma_1}(f)$ and $\pi_{\Gamma_2}(f)$ can generate the face f . Now let X_1 and X_2 be the simple complexes with exactly those faces from the reductive projections of X . By Proposition 4.10, $X_1 \otimes X_2$ is a simple complex, and in particular $X_1 \otimes X_2$ has no duplicated faces. By the assumption of (2), X contains all the faces of $X_1 \otimes X_2$. Conversely, any face f of X is a face of $X_1 \otimes X_2$, since f can be generated by $\pi_{\Gamma_1}(f)$ and $\pi_{\Gamma_2}(f)$. Then we have $X = X_1 \otimes X_2$, and hence (2) implies (1). \square

Although we already know the associativity of complex tensor product through the universal property, it will be helpful to understand how faces are formed in the product of more than two complexes. First let us review the product of two complexes. Let f_α be a face of length n_α attached along a cycle C_α in X , and f_β be a face of length n_β

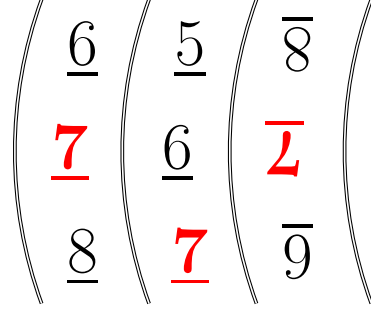


Figure 14: a face generated by 3 faces in complex tensor product

attached along a cycle C_β in Y . By Definition 3.1, f_α and f_β generate faces $f_{\alpha,\beta}^{i_\delta}$ of length $[n_\alpha, n_\beta]$ attached along $(C_\alpha, C_\beta)_{\otimes}^{i_\delta}$, $i \in \{0, 1, \dots, (n_\alpha, n_\beta) - 1\}$, $\delta \in \{0, 1\}$. To explain the boundary cycle of $f_{\alpha,\beta}^{i_\delta}$ in plain language, basically we pick a pair of corners of f_α and f_β to start, and go around C_α and C_β in two coordinates respectively until we return to the starting pair of corners. Note that the index i is chosen in such a way that each pair of corners appears exactly once among all faces generated by f_α and f_β .

A good way to visualize this is a slot machine of two reels of length $[n_\alpha, n_\beta]$, cyclically labeled by the vertices of f_α and f_β respectively. Faces generated by f_α and f_β have a one-to-one correspondence with different combinations of two reels, with flipping allowed for the second reel. From this aspect, it is easy to see that for face f_j of length n_j in complex X_j , $j \in \{1, 2, \dots, m\}$, f_1, f_2, \dots, f_m generate faces in $\otimes_{j=1}^m X_j$ of length $[n_1, n_2, \dots, n_m]$ such that each m -tuple of corners appears exactly once among all generated faces. Faces generated by f_1, f_2, \dots, f_m have a one-to-one correspondence with different combinations of m reels of length $[n_1, n_2, \dots, n_m]$, cyclically labeled by the vertices of f_j respectively, with flipping allowed from the second reel on. Figure 14 illustrates how a face is generated by the complex tensor product of 3 faces from such an aspect.

Theorem 4.12. Let X be a simple polygonal cell complex. If the 1-skeleton of X is a finite simple connected non-bipartite R -thin edge-transitive graph with more than one vertex, then X has a unique factorization into prime complexes.

Proof. By Theorem 4.4, since $X^1 \in \mathfrak{S} \subset \mathfrak{S}_0$ is a finite connected non-bipartite graph with more than one vertex, X^1 has a unique factorization $X^1 = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$ into primes in \mathfrak{S}_0 with respect to direct product of graphs. By Proposition 4.6, the edge-transitivity of X^1 implies that each prime factor Γ_i is in fact a simple graph. On the other hand, if we factorize X^1 with respect to graph tensor product, each factor would also be a simple graph with more than one vertex, because a loop creates double edges in the product, and a single vertex breaks the connectivity of the product. Note that direct product and graph tensor product coincide in \mathfrak{S} . Hence we know X^1 has a unique factorization $X^1 = \Gamma_1 \otimes \Gamma_2 \otimes \dots \otimes \Gamma_n$ into primes in \mathfrak{S} with respect to graph tensor product.

Now we consider the factorization of the complex X . Note that we can always obtain a prime factorization of X , since the number of vertices of factors decreases as the factoring goes. Suppose X has two factorizations A and B , and X_0 is a prime factor of X in A with 1-skeleton $\Gamma_1 \otimes \Gamma_2$. By Proposition 4.11, there exist two faces f_1 and f_2 such that X_0 lacks certain face generated by $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$. In other words, there is certain

pair of corners of $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$ missing in the faces of X_0 , and hence such pair will be absent in the n -tuples representing face corners of X . By Proposition 4.9, we can find faces $\overline{f_1}$ and $\overline{f_2}$ of X such that $\pi_{\Gamma_1 \otimes \Gamma_2}(\overline{f_1}) = f_1$ and $\pi_{\Gamma_1 \otimes \Gamma_2}(\overline{f_2}) = f_2$, and we have $\pi_{\Gamma_1}(\overline{f_1}) = \pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(\overline{f_2}) = \pi_{\Gamma_2}(f_2)$. If Γ_1 and Γ_2 belong to different prime factors X_1 and X_2 in B , we can reductively project $\overline{f_i}$ to X_i to obtain a face f'_i of X_i , $i \in \{1, 2\}$. Then we have $\pi_{\Gamma_1}(f'_1) = \pi_{\Gamma_1}(\overline{f_1}) = \pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f'_2) = \pi_{\Gamma_2}(\overline{f_2}) = \pi_{\Gamma_2}(f_2)$. Notice that f'_1 and f'_2 generate all possible pairs of corners of $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$ in $X_1 \otimes X_2$ and hence in X , a contradiction. So Γ_1 and Γ_2 belong to the same prime factor in B .

The above argument can be applied to the case when the 1-skeleton of X_0 is the graph tensor product of more than two prime graphs, simply by splitting prime graph factors into two groups. It follows that every prime 1-skeleton factor of X_0 belongs to the same prime complex X'_0 in B . Conversely, every prime 1-skeleton factor of X'_0 belongs to X_0 , and hence X_0 and X'_0 are actually the same. In case X_0 has a prime 1-skeleton Γ_j , then Γ_j belongs to some X'_0 in B with a prime 1-skeleton, otherwise the prime 1-skeleton factors of X'_0 belong to at least two complexes in A . In conclusion, we know two factorizations A and B are identical, and X has a unique factorization into prime complexes. \square

Theorem 4.13. Suppose that X is a simple polygonal cell complex, and its 1-skeleton is a finite simple connected non-bipartite edge-transitive R -thin graph with more than one vertex. Let $X = X_1 \otimes X_2 \otimes \cdots \otimes X_n$ be a prime factorization of X . Then $\text{Aut}(X)$ is generated by automorphisms of prime factors and permutations of isomorphic factors.

Proof. Since X has no faces attached along the same cycle, an automorphism of X is completely determined by its action on the 1-skeleton X^1 , and we can identify $\text{Aut}(X)$ as a subgroup of $\text{Aut}(X^1)$. To understand $\text{Aut}(X^1)$, by the argument in the proof of Theorem 4.12, we know X^1 has a unique factorization $X^1 = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_m = \Gamma_1 \otimes \Gamma_2 \otimes \cdots \otimes \Gamma_m$ into primes in \mathfrak{S} . By Theorem 4.5, the extra R -thin condition on X^1 implies that $\text{Aut}(X^1)$ is generated by automorphisms of Γ_i 's and permutations of isomorphic Γ_j 's.

Let φ be an arbitrary automorphism of X , which can be represented as some $\rho \in \times_{i=1}^m \text{Aut}(\Gamma_i)$ followed by a permutation of Γ_j 's. This implies that for any face f of X

$$\varphi(\pi_{\otimes_{i \in I} \Gamma_i}(f)) = \pi_{\varphi(\otimes_{i \in I} \Gamma_i)}(\varphi(f)) = \pi_{\otimes_{i \in I} \varphi(\Gamma_i)}(\varphi(f)),$$

where I is an arbitrary non-empty subset of $\{1, 2, \dots, m\}$. Suppose that X_1 has 1-skeleton $X_1^1 = \otimes_{i \in I} \Gamma_i$ for some $I \subset \{1, 2, \dots, m\}$. We claim that $\forall i \in I$, $\varphi(\Gamma_i)$ belongs to the same prime factor X_k of X . If not, then we can find $I_1 \sqcup I_2 = I$ such that $\forall i \in I_1, \forall j \in I_2$, $\varphi(\Gamma_i)$ and $\varphi(\Gamma_j)$ belong to different prime factors of X . Let $\Gamma_\alpha = \otimes_{i \in I_1} \Gamma_i$ and $\Gamma_\beta = \otimes_{j \in I_2} \Gamma_j$, and hence we have $X_1^1 = \Gamma_\alpha \otimes \Gamma_\beta$. Since X_1 is prime, by Proposition 4.11, we can find faces f_1 and f_2 of X_1 such that X_1 lacks certain face generated by $\pi_{\Gamma_\alpha}(f_1)$ and $\pi_{\Gamma_\beta}(f_2)$. By Proposition 4.9, we can find faces $\overline{f_1}$ and $\overline{f_2}$ of X such that $\pi_{\Gamma_\alpha \otimes \Gamma_\beta}(\overline{f_1}) = f_1$ and $\pi_{\Gamma_\alpha \otimes \Gamma_\beta}(\overline{f_2}) = f_2$. Then the complex X lacks certain corner combination of $\pi_{\Gamma_\alpha}(\overline{f_1})$ and $\pi_{\Gamma_\beta}(\overline{f_2})$ in the m -tuples representing face corners of X . By taking the automorphism φ , the complex X lacks certain corner combination of $\pi_{\otimes_{i \in I_1} \varphi(\Gamma_i)}(\varphi(\overline{f_1}))$ and $\pi_{\otimes_{j \in I_2} \varphi(\Gamma_j)}(\varphi(\overline{f_2}))$, which is impossible because $\varphi(\Gamma_i)$ and $\varphi(\Gamma_j)$ belong to different prime factors of X , and taking complex tensor product of these factors generates all the corner combinations.

Hence for every 1-skeleton factor Γ_i of X_1 , $\varphi(\Gamma_i)$ belongs to the same prime factor X_k of X . By considering φ^{-1} , we know that X_k has exactly these $\varphi(\Gamma_i)$'s as 1-skeleton

factors. Moreover, $\varphi(\pi_{\otimes_{i \in I} \Gamma_i}(f)) = \pi_{\otimes_{i \in I} \varphi(\Gamma_i)}(\varphi(f))$ implies that φ induces an isomorphism from X_1 to X_k . This shows that every $\varphi \in \text{Aut}(X)$ can be represented as some $\sigma \in \times_{i=1}^n \text{Aut}(X_i)$ followed by a permutation of X_j 's, and the theorem holds. \square

Remark. Let \tilde{X} be the disjoint union of prime factors of X . Then the above theorem implies that $\text{Aut}(X) \cong \text{Aut}(\tilde{X})$, which is a convenient way to describe $\text{Aut}(X)$.

The following corollary is a partial converse of Theorem 3.10.

Corollary 4.14. Suppose that X is a simple polygonal cell complex, and its 1-skeleton is a finite simple connected non-bipartite edge-transitive R -thin graph with more than one vertex. If X is flag-transitive, then any factor of X is flag-transitive.

Proof. Note that it suffices to show that any prime factor of X is flag-transitive. Then by Theorem 4.12 and Theorem 3.10, any factor of X is a complex tensor product of flag-transitive prime factors of X , and hence is flag-transitive.

By Theorem 4.12, X has a unique prime factorization $X = X_1 \otimes X_2 \otimes \cdots \otimes X_n$. Suppose that one of the prime factors is not flag-transitive, without loss of generality say X_1 , and X_i is isomorphic to X_1 if and only if $1 \leq i \leq m$ for some integer $m \leq n$. Since X_1 is not flag-transitive, there exist two oriented face corners (e_1^1, v_1, e_1^2) and $(e_1^{1'}, v_1', e_1^{2'})$ in X_1 such that $\text{Aut}(X_1)$ can not map one corner to the other. For each j such that $m+1 \leq j \leq n$, we pick an arbitrary corner (e_j^1, v_j, e_j^2) of X_j . Consider the following two corners of X :

$$\begin{aligned} &((e_1^1, \dots, e_1^1, e_{m+1}^1, \dots, e_n^1), (v_1, \dots, v_1, v_{m+1}, \dots, v_n), (e_1^2, \dots, e_1^2, e_{m+1}^2, \dots, e_n^2)) \text{ and} \\ &((e_1^{1'}, \dots, e_1^{1'}, e_{m+1}^1, \dots, e_n^1), (v_1', \dots, v_1', v_{m+1}, \dots, v_n), (e_1^{2'}, \dots, e_1^{2'}, e_{m+1}^2, \dots, e_n^2)). \end{aligned}$$

By Theorem 4.13, $\text{Aut}(X)$ is generated by automorphisms of prime factors and permutation of isomorphic factors. In particular, it is impossible for $\text{Aut}(X)$ to map one of the above corners to the other, contradicting to the flag-transitivity of X . Therefore we can conclude that any prime factor of X is flag-transitive. \square

The corollary below answers the question we posed in the beginning of the chapter.

Corollary 4.15. For $i \in \{1, 2, \dots, n\}$, let X_i be a simple prime complex with a finite simple connected non-bipartite symmetric R -thin 1-skeleton having more than one vertex. Then the complex tensor product $X = \otimes_{i=1}^n X_i$ has automorphism group $\text{Aut}(X)$ generated by $\text{Aut}(X_i)$'s and permutations of isomorphic X_j 's.

Proof. By Proposition 4.10, we know X is a simple complex. By the definition of graph tensor product, we know X^1 is a finite simple graph. Note that a simple graph is non-bipartite if and only if there is a cycle of odd length. Then the graph tensor product of two non-bipartite graphs contains a cycle of odd length and hence is non-bipartite. Induction shows that X^1 is non-bipartite, and by Theorem 4.3 we know that X^1 is connected. By the special case of Theorem 3.10 (complexes without faces), we know X^1 is symmetric and hence edge-transitive. Note that for two graphs Γ_1 and Γ_2 , the set of neighbours of a vertex $(u, v) \in V(\Gamma_1 \otimes \Gamma_2)$ is the direct product of the set of neighbours of u in Γ_1 with the set of neighbours of v in Γ_2 . This implies the graph tensor product of R -thin graphs is a R -thin graph. To summarize, we know X is a simple complex with a prime factorization $X = \otimes_{i=1}^n X_i$, and its 1-skeleton X^1 is a finite simple connected non-bipartite edge-transitive R -thin graph with more than one vertex. By Theorem 4.13, we know that $\text{Aut}(X)$ is as described in the corollary. \square

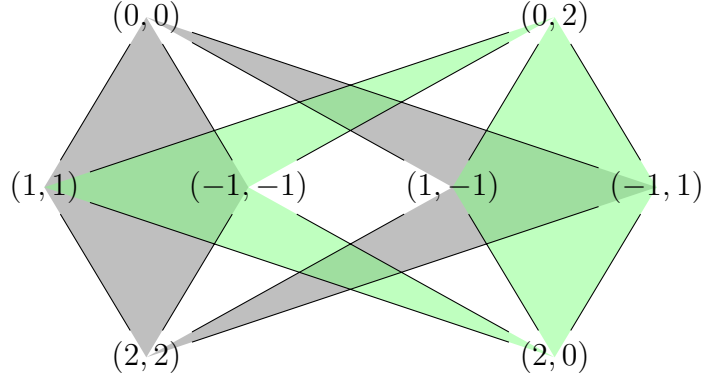


Figure 15: a component of the tensor product of two squares

Remark. The tensor products of edge-transitive graphs are not necessarily edge-transitive. Therefore we require each X_i^1 to be symmetric to ensure the edge-transitivity of X^1 .

Note that when a complex has a face of odd length, then the 1-skeleton of the complex is non-bipartite, and Corollary 4.15 has a chance to work. In the next chapter, we will investigate the automorphism group of the tensor product of complexes with only faces of even lengths from a different aspect.

5 Even Cases

In this chapter we investigate the tensor product of complexes with only faces of even lengths, and our goal is to develop results similar to Corollary 4.15, which basically says an automorphism of certain complex tensor products must be of Cartesian type. Note that when there is more than one bipartite factor, Theorem 4.3 implies that the complex tensor product is disconnected, and the product is likely to have non-Cartesian automorphisms from the direct product of automorphism groups of components. Hence in such a context, the proper question to pose should be as follows: for complexes X_i with only faces of even lengths, is the automorphism group of a component of $\otimes X_i$ generated by automorphisms of X_i 's together with permutations of isomorphic factors?

For graph tensor products, the connectedness of the product does not guarantee the absence of non-Cartesian automorphisms. For complex tensor products, we hope that the extra face structure helps to eliminate non-Cartesian automorphisms. For example, let us look at the complex tensor product of two squares, which has two isomorphic components. We denote vertices of a square by $0, 1, 2, -1$ cyclically, and illustrate one component of the product in Figure 15. Note that the 1-skeleton of the component is actually a complete bipartite graph with $2 \cdot 4! \cdot 4!$ automorphisms, and not all of them give a complex automorphism due to the extra face structure.

Figure 15 also reveals an important fact of the tensor product of complexes with only faces of even lengths: a face is antipodally attached to another face generated by the same pair of faces, and through such antipodally attached relation we can find all other faces generated by the same pair of faces in that component. Such face blocks (defined in Definition 5.4) help to determine the Cartesian structure of a complex tensor product, and if we can show a generic face block has only Cartesian automorphisms, then we have a

chance to force a complex automorphism stabilizing a face block to be of Cartesian type. To simplify the problem, we restrict our discussion to the tensor product of complexes with faces of the same even length, and the first step is to establish the Cartesian result for the tensor product of $2n$ -gons. The following lemma is a useful tool for this purpose.

Lemma 5.1. Suppose on a real line, someone wants to take d steps to walk from an integer $d - 2k$ to 0, where $\lfloor \frac{d}{2} \rfloor \geq k \geq 0$ is an integer, and each step is either plus 1 or minus 1. Then there are $\binom{d-1}{k}$ ways to arrive from 1, and $\binom{d-1}{k-1}$ ways to arrive from -1 . The ratio $\binom{d-1}{k} / \binom{d-1}{k-1}$ is greater than or equal to 1, with equality if and only if $d - 2k = 0$. Moreover, when d is fixed and k is increasing, the ratio is decreasing.

Proof. Suppose this person takes x steps of minus 1 and y steps of plus 1 to arrive at 0. Then we have $x + y = d$ and $-x + y = -d + 2k$, and therefore $x = d - k$ and $y = k$. By ordering two types of steps arbitrarily, we can obtain all different ways to arrive at 0. To arrive from 1, the last step must be minus 1, and there are $\binom{d-1}{k}$ such combinations. To arrive from -1 , the last step must be plus 1, and there are $\binom{d-1}{k-1}$ such combinations. When d is odd, we have $\frac{d-1}{2} \geq k$ and hence $\binom{d-1}{k} > \binom{d-1}{k-1}$. When d is even, we have $\frac{d}{2} \geq k$ which implies $\frac{d-1}{2} > k - 1$ and hence $\binom{d-1}{k} \geq \binom{d-1}{k-1}$, with equality if and only if $k + (k - 1) = d - 1$, namely $d - 2k = 0$. To show that the ratio $\binom{d-1}{k} / \binom{d-1}{k-1}$ decreases as k increases, we simply have to verify the following inequality:

$$\begin{aligned}
& \binom{d-1}{k} / \binom{d-1}{k-1} > \binom{d-1}{k+1} / \binom{d-1}{k} \\
\Leftrightarrow & \binom{d-1}{k} \binom{d-1}{k} > \binom{d-1}{k+1} \binom{d-1}{k-1} \\
\Leftrightarrow & \frac{(d-1) \dots (d-k)}{k!} \frac{(d-1) \dots (d-k)}{k!} > \frac{(d-1) \dots (d-k-1)}{(k+1)!} \frac{(d-1) \dots (d-k+1)}{(k-1)!} \\
\Leftrightarrow & \frac{d-k}{k} > \frac{d-k-1}{k+1} \\
\Leftrightarrow & \frac{d}{k} - 1 > \frac{d}{k+1} - 1 \\
\Leftrightarrow & k + 1 > k,
\end{aligned}$$

which is obviously true. □

Proposition 5.2. For $i \in \{1, 2, \dots, m\}$, let C_i be a graph which is a cycle of length $2n$, where n is an integer at least 3. Then the automorphism group of a component of $\otimes_{i=1}^m C_i$ can be generated by elements of $\text{Aut}(C_i)$'s together with permutations of C_i 's.

Proof. We denote vertices of C_i by $0, 1, \dots, n-1, n, -(n-1), -(n-2), \dots, -1$ cyclically, and let Γ be the component of $\otimes_{i=1}^m C_i$ containing the vertex $v = (0, 0, \dots, 0)$. Note that $\times_{i=1}^m \text{Aut}(C_i)$ acts transitively on vertices of $\otimes_{i=1}^m C_i$. Therefore to prove this proposition, it suffices to show that the v -stabilizer G_v of $\text{Aut}(\Gamma)$ can be generated by elements of $\text{Aut}(C_i)$'s together with permutations of C_i 's. Notice that there are $2^m \cdot m!$ Cartesian automorphisms of Γ fixing v , generated by the reflection fixing 0 in each C_i and all permutations of m factors. If we can show $|G_v| \leq 2^m \cdot m!$, then the proposition follows.

First we show that Γ is a rigid graph. Namely we want to show that if $\varphi \in G_v$ fixes all neighbours of v , then φ must be trivial. Note that two vertices (b_1, b_2, \dots, b_m) and (c_1, c_2, \dots, c_m) are adjacent if and only if $b_i - c_i \equiv \pm 1 \pmod{2n}$ for all i , and therefore

$$V(\Gamma) \subseteq V^* = \{(a_1, a_2, \dots, a_m) \in V(\otimes_{i=1}^m C_i) \mid a_1 \equiv a_2 \equiv \dots \equiv a_m \pmod{2}\}.$$

For each $u = (a_1, a_2, \dots, a_m) \in V^*$, there is a path of length $d = \max\{|a_1|, |a_2|, \dots, |a_m|\}$ from u to v , because we can reach 0 in d steps in the coordinates with absolute value d , and we can also reach 0 in d steps in the other coordinates by walking back and forth as each coordinate has the same parity. Hence $V(\Gamma) = V^*$, and $d(u, v) = d$ follows easily.

Note that the number of geodesics from u to v is the product of the number of ways in each coordinate to walk to 0 in d steps. Look at the i -th coordinate of v . For now we assume that $a_i \geq 0$, and let k_i be the integer such that $a_i = d - 2k_i$. If $n > a_i > 0$, we have $\lfloor \frac{d}{2} \rfloor \geq k_i \geq 0$, and walking to 0 in d steps is equivalent to the setting of Lemma 5.1. By the lemma, the ratio of numbers of $u - v$ geodesics arriving from 1 and from -1 in the i -th coordinate is $\binom{d-1}{k_i} / \binom{d-1}{k_i-1} > 1$. Since the automorphism φ fixes $(\pm 1, \pm 1, \dots, \pm 1)$ and preserves geodesics, this ratio does not change under φ . Again by the Lemma, k_i must remain the same to keep this ratio, and hence the i -th coordinate of $\varphi(u)$ must be a_i . If $a_i = 0$, then u has a neighbour w with the i -th coordinate 1. Note that $\varphi(u)$ is adjacent to $\varphi(w)$ with the i -th coordinate 1, and the i -th coordinate of $\varphi(u)$ is either 0 or 2. In the latter case, since $n > 2 > 0$, by taking φ^{-1} the above argument implies $a_i = 2$, a contradiction. Hence the i -th coordinate of $\varphi(u)$ is 0. Similarly if $a_i = n$, then the i -th coordinate of $\varphi(u)$ is n . For negative a_i , by applying the mirror version of Lemma 5.1, we know that the i -th coordinate of $\varphi(u)$ is a_i . Note that the above result is true for every coordinate. Hence $\varphi(u) = u$ for every $u \in V(\Gamma)$, and φ is trivial.

Now look at the local structure around v . Note that two neighbours of v taking different values in k coordinates have 2^{m-k} common neighbours. In particular, two neighbours of v differ in exactly one coordinate if and only if they have 2^{m-1} common neighbours. Hence among the neighbours of v , the relation of differing in exactly one coordinate is preserved under G_v . If we draw an auxiliary edge between any two such neighbours of v , then the 2^m neighbours of v plus these auxiliary edges form a hypercube Q_m preserved under G_v . Since Γ is rigid, an automorphism of G_v is completely determined by its action on the neighbours of v , which also induces an automorphism of the auxiliary Q_m . As a result, we have $|G_v| \leq \text{Aut}(Q_m) = 2! \cdot m!$, which finishes the proof. \square

Remark. Let H be the subgroup of $\times_{i=1}^m \mathbb{Z}_{2n}$ generated by $S = \{(\pm 1, \pm 1, \dots, \pm 1)\}$. Note that the component Γ in the above proof is actually isomorphic to the Cayley graph of H with respect to the generating set S .

Corollary 5.3. Suppose that X_i is a $2n$ -gon for $i \in \{1, 2, \dots, m\}$, where n is an integer at least 3. Then the automorphism group of a component of $\otimes_{i=1}^m X_i$ can be generated by elements of $\text{Aut}(X_i)$'s together with permutations of X_i 's.

Proof. Note that a $2n$ -gon has the same automorphism group as its 1-skeleton, and $\otimes_{i=1}^m X_i$ has the same Cartesian automorphisms as $\otimes_{i=1}^m X_i^1$. Hence a vertex stabilizer G_v of a component X of $\otimes_{i=1}^m X_i$ has $2^m \cdot m!$ Cartesian automorphisms, and $|G_v|$ is at most the cardinality of the stabilizer of v in X^1 , which is $2^m \cdot m!$ by Proposition 5.2. \square

Remark. We do need the condition $n \geq 3$ in Proposition 5.2 and Corollary 5.3. For $n = 2$, Figure 15 illustrates a component of the tensor product of two squares. Its 1-skeleton is the complete bipartite graph $K_{4,4}$ with lots of non-Cartesian automorphisms. With the face structure, there are much fewer complex automorphisms, but swapping $(0, 2)$ and $(2, 0)$ still gives a non-Cartesian complex automorphism.

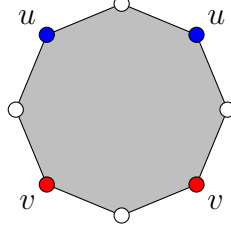


Figure 16: a non-elementary complex

Now we formally define the face blocks mentioned in the beginning of the chapter. An intuitive definition of a face block in a complex tensor product $\otimes_{i=1}^m X_i$ would be any connected component in $\otimes_{i=1}^m f_i$, where each f_i is a face of X_i . Note that if each f_i is an even gon attached injectively, then $\otimes_{i=1}^m f_i$ has 2^{m-1} components, and hence 2^{m-1} face blocks. If these f_i 's are attached non-injectively, then the above face blocks could have extra incidence relations, and we might end up having fewer components. We would like to define a face block regardless of attaching maps, so we take the following definition.

Definition 5.4. For $i \in \{1, 2, \dots, m\}$, let X_i be a polygonal cell complex with only faces of even length $2n \geq 2$. Let f_i be a face of X_i with corners labeled by $0, 1, \dots, 2n-1$ cyclically. A **face block** generated by f_1, f_2, \dots, f_m is a subcollection of faces generated by f_1, f_2, \dots, f_m such that two faces f_a and f_b are in the same face block if and only if a corner of f_a with label (a_1, a_2, \dots, a_m) and a corner f_b with label (b_1, b_2, \dots, b_m) have

$$a_1 - b_1 \equiv a_2 - b_2 \equiv \dots \equiv a_m - b_m \pmod{2}.$$

Remark. It is easy to see that a face block is well-defined no matter how faces are cyclically labeled and no matter which corners are chosen to verify the above criterion. In general it is not obvious whether or not two faces are in the same face block of a complex tensor product without knowing the tensor product structure. In the tensor product of the following class of complexes, recognizing a face block is much easier.

Definition 5.5. A connected polygonal cell complex X is an **elementary** complex if X satisfies the following three conditions:

- (1) Every face of X is of the same even length ≥ 2 .
- (2) No antipodal corners of a face are attached to the same vertex.
- (3) For any two vertices, there is at most one pair of antipodal face corners attached.

Remark. Condition (3) basically says no two faces can be attached antipodally, and in a face different pairs of antipodal corners are not attached to the same pair of vertices. For example, the complex in Figure 16 is not an elementary complex.

Proposition 5.6. For $i \in \{1, 2, \dots, m\}$, let X_i be an elementary complex with faces of even length $2n \geq 2$. Then in the complex tensor product $\otimes_{i=1}^m X_i$, for any antipodal vertices u and v of a face in $\otimes_{i=1}^m X_i$, there are exactly 2^{m-1} faces having u and v as antipodal vertices, and these faces are in the same face block. Moreover, for any two faces f and f' in the same face block, we can find a series of faces f_0, f_1, \dots, f_k such that $f_0 = f$, $f_k = f'$, f_i and f_{i+1} share antipodal vertices for $i \in \{0, 1, \dots, k-1\}$, and $k \leq n$.

Proof. In $\otimes_{i=1}^m X_i$, suppose that $u = (u_1, u_2, \dots, u_m)$ and $v = (v_1, v_2, \dots, v_m)$ are antipodal vertices of a face f generated by f_1, f_2, \dots, f_m , where u_i and v_i are vertices of X_i and f_i is a face of X_i for $i \in \{1, 2, \dots, m\}$. Note that for each $i \in \{1, 2, \dots, m\}$, projecting f to X_i gives f_i , and f_i has u_i and v_i as antipodal vertices. Since X_i is elementary, u_i and v_i are not the same vertex, and f_i is the only face of X_i having u_i and v_i as antipodal vertices, with a unique pair of antipodal corners attached to u_i and v_i . Hence any face in $\otimes_{i=1}^m X_i$ having u and v as antipodal vertices must be generated by f_1, f_2, \dots, f_m in such a way that the corresponding corners c_i of the f_i 's at u_i are combined together. With the corner c_1 of f_1 fixed, flipping f_i at c_i for $i \in \{1, 2, \dots, m\}$ gives all 2^{m-1} faces having u and v as antipodal vertices, and these faces are in the same face block.

Now suppose that f and f' are two faces in the same face block B generated by faces with corners labeled by $0, 1, \dots, 2n - 1$ cyclically. Then we can label corners in B according to such a corner labeling, and by following steps of $(\pm 1, \pm 1, \dots, \pm 1)$, we can start from a vertex v of f to reach any other vertex in B in n steps. In particular, there is a unique vertex in B such that we need n steps to reach it from v . Since f' has more than one vertex, we can start from v to reach a vertex u of f' in $n - 1$ steps. By adding one step in f and one step in f' if necessary, we can find a path from f to f' of length at most $n + 1$ such that the first and the last steps are in f and f' respectively. Note that each $(\pm 1, \pm 1, \dots, \pm 1)$ step determines a unique face in B , and hence the above path determines a series of faces f_0, f_1, \dots, f_k such that $f_0 = f$, $f_k = f'$, and $k \leq n$. If f_i and f_{i+1} are determined by the same $(\pm 1, \pm 1, \dots, \pm 1)$ step, then f_i and f_{i+1} are actually the same face, and we can remove one of them from the sequence. If f_i and f_{i+1} are determined by different $(\pm 1, \pm 1, \dots, \pm 1)$ steps, then f_i and f_{i+1} are two different faces with a common vertex with label (a_1, a_2, \dots, a_m) . Note that

$$(a_1, a_2, \dots, a_m) + n(\pm 1, \pm 1, \dots, \pm 1) = (a_1 + n, a_2 + n, \dots, a_m + n) \pmod{2n},$$

which is also a common vertex of f_i and f_{i+1} , and therefore f_i and f_{i+1} share antipodal vertices. The above argument is illustrated in Figure 15. \square

Proposition 5.6 allows us to easily recognize a face block in a complex tensor product. If we impose the following conditions on each factor, then we can read the Cartesian structure of a complex tensor product through the incidence relation of face blocks.

Definition 5.7. A connected polygonal cell complex X is an **ordinary** complex if every face f of X is of the same even length $2n \geq 4$, and satisfies the following extra conditions:

- (1) If we label corners of f cyclically from 1 to $2n$, then any two corners with different parities are not attached to the same vertex.
- (2) For any face f' incident to f , either f has only one corner meeting f' , or f has only two consecutive corners meeting f' .

Remark. If the 1-skeleton of X is bipartite, then X satisfies (1) automatically. Also note that a polygonal complex satisfies both (1) and (2). The reader might have noticed that (2) implies the condition (3) of an elementary complex. Since there are alternative conditions serving our purpose as effectively as (2), we avoid defining ordinary complexes as a subclass of elementary complexes.

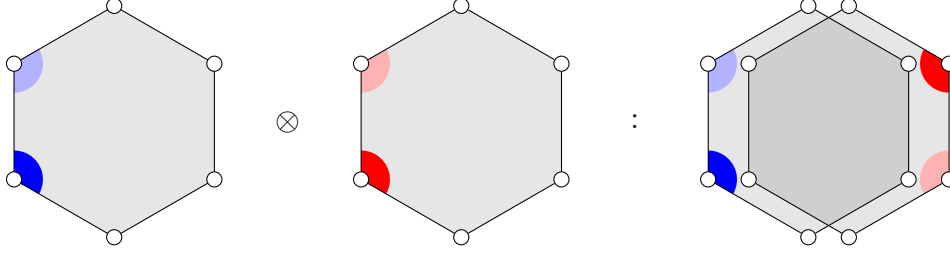


Figure 17: how to avoid incidence

Proposition 5.8. For $i \in \{1, 2, \dots, m\}$, suppose that X_i is an ordinary complex with faces of even length $2n \geq 4$. Let B be a face block generated by f_1, f_2, \dots, f_m and B' be a face block generated by f'_1, f'_2, \dots, f'_m , where f_i and f'_i are faces of X_i . If B and B' are incident, then the following two statements are equivalent:

- (1) $\exists j$ such that f_j is incident to f'_j in X_j , and $\forall i \neq j$ we have $f_i = f'_i$.
- (2) Every face of B is incident to a face of B' .

Proof. Assume (1). Without loss of generality, we can assume that $j = 1$. Since B and B' are incident, there is a face corner c of B meeting a face corner c' of B' . Suppose that c is the combination of corners c_i of the f_i 's, and c' is the combination of corners c'_i of the f'_i 's. Note that c_1 of f_1 meets c'_1 of f'_1 in X_1 . Also note that for $i \neq 1$, c_i and c'_i are in the same face f , and they are either the same corner or different corners attached to the same vertex. In particular, by condition (1) of Definition 5.7, c_i and c'_i have the same parity under cyclic \mathbb{Z}_{2n} labeling for $i \neq 1$. Let f be an arbitrary face of B generated by combining c_1 of f_1 with corners \bar{c}_i of the f_i 's for $i \neq 1$. By Definition 5.4, \bar{c}_i has the same parity as c_i , and therefore has the same parity as c'_i . Then again by Definition 5.4, the face f' generated by combining c'_1 of f'_1 with \bar{c}_i 's of the f_i 's is a face of B' . It is obvious that f is incident to f' . To summarize, given an arbitrary face f of B , we can find a face f' of B' incident to f . Hence (1) implies (2).

Assume (2). If f_i and f'_i are disjoint, then B and B' are disjoint, which contradicts (2). Hence for each $i \in \{1, 2, \dots, m\}$, f_i and f'_i are either incident or actually the same. Suppose that there is more than one j , say for $j \in \{1, 2\}$, such that f_j and f'_j are incident. By condition (2) of Definition 5.7, f_1 and f_2 have either one corner or two consecutive corners meeting f'_1 and f'_2 respectively. Pick two consecutive corners of f_1 containing all corners meeting f'_1 and colour them blue. Similarly pick two consecutive corners of f_2 containing all corners meeting f'_2 and colour them red. Consider the faces generated by f_1, f_2, \dots, f_m with the following corner combination: coloured corners of f_1 and f_2 are placed at the opposite positions, as illustrated in Figure 17. Note that these faces are disjoint with faces generated by f'_1, f'_2, \dots, f'_m . If B does not contain any of these faces, we can flip two red corners of f_2 to generate faces of B , and the resulting faces are still disjoint with faces generated by f'_1, f'_2, \dots, f'_m . In other words, we can find a face of B incident to no face in B' , a contradiction. So there is at most one j such that f_j and f'_j are incident. Moreover, condition (1) of Definition 5.7 implies that different face blocks generated by f_1, f_2, \dots, f_m are disjoint. Since B and B' are incident, we know that there is exactly one j such that f_j and f'_j are incident. Hence (2) implies (1). \square

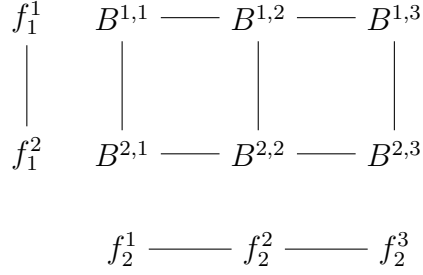


Figure 18: Cartesian structure of face blocks

Remark. Note that condition (2) of Definition 5.7 is only used for the argument illustrated in Figure 17. It is not hard to have alternative conditions serving this purpose, especially when the length of faces is higher. We also want to point out that through finer examination of incidence relation between face blocks, it is possible to obtain more information such as how f_j meets f'_j in X_j , perhaps under weaker conditions.

With Propositions 5.6 and 5.8, in a tensor product $X = X_1 \otimes X_2 \otimes \cdots \otimes X_m$ where each X_i is an elementary ordinary complex with only faces of even length $2n \geq 4$, we can recognize face blocks and the Cartesian structure of X through the incidence relation on faces, which is preserved under automorphisms of X . Now we define a graph Γ_X to encode the Cartesian structure of X . Let Γ_X be a simple graph with vertex set $\times_{i=1}^m F(X_i)$, where a vertex (f_1, f_2, \dots, f_m) represents all faces of X generated by f_1, f_2, \dots, f_m , such that two vertices are adjacent if and only if they take the same face in $m-1$ coordinates, and have incident faces in the remaining coordinate. Let Γ_{X_i} be a simple graph with vertex set $F(X_i)$, such that two vertices are adjacent if and only if the corresponding faces are incident in X_i . Notice that $\Gamma_X = \Gamma_{X_1} \square \Gamma_{X_2} \square \cdots \square \Gamma_{X_m}$, where \square is the Cartesian product of graphs (see [4] for the definition). Figure 18 illustrates the case $m=2$, where $B^{i,j} = (f_1^i, f_2^j)$ represents all faces generated by f_1^i and f_2^j . The following theorem due to Imrich [5] and Miller [7] restricts the automorphism group of Γ_X .

Theorem 5.9. Suppose that Γ is a finite simple connected graph with a factorization $\Gamma = \Gamma_1 \square \Gamma_2 \square \cdots \square \Gamma_m$, where each Γ_i is prime with respect to Cartesian product. Then the automorphism group of Γ is generated by automorphisms of prime factors and permutations of isomorphic factors.

We can not guarantee Γ_{X_i} is prime, but at least X_i is indeed a prime complex.

Proposition 5.10. Let Y be an elementary complex. Then Y is a prime with respect to complex tensor product, and Y is not a component of any complex tensor product.

Proof. Suppose that there exist complexes Y_1 and Y_2 such that Y is a component of $Y_1 \otimes Y_2$. Note that a face of Y is of even length, and must be generated by either two even faces or by one even and one odd face. In either case, by Definition 3.1, Y will have faces antipodally attached together, violating that Y is elementary. \square

Note that in Figure 18, each $B^{i,j}$ actually contains two face blocks generated by f_1^i and f_2^j , and in general each vertex of Γ_X defined above contains 2^{m-1} face blocks. Even if we have some control over the automorphism group of Γ_X , having multiple face blocks

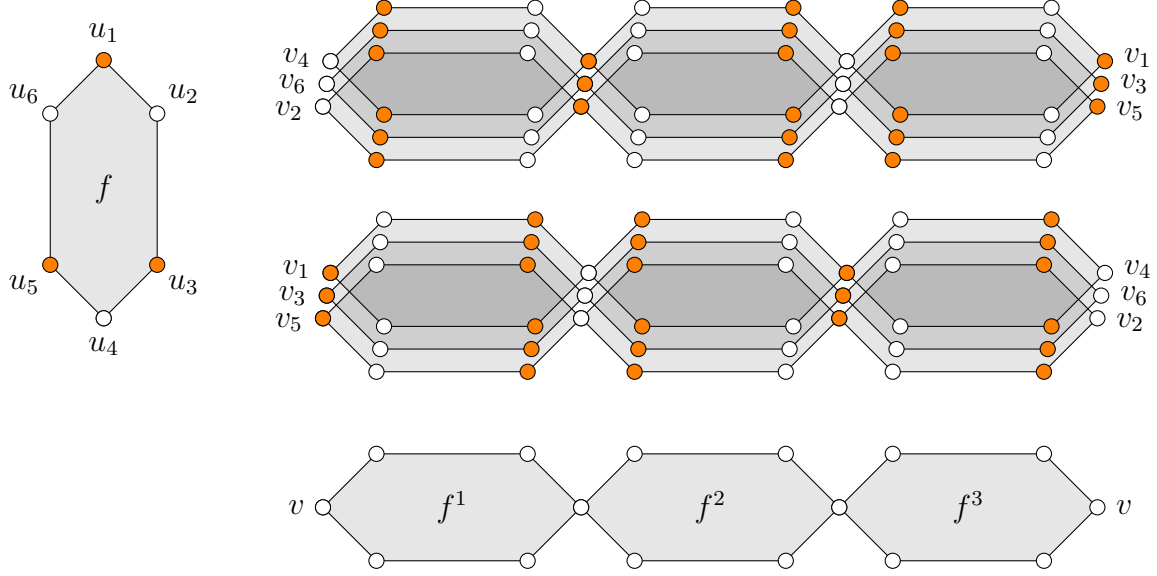


Figure 19: tensor product of a hexagon with a 3-hexagon necklace

at one vertex of Γ_X could lead to non-Cartesian automorphisms of X . Let us look at the tensor product of a hexagon with a 3-hexagon necklace as illustrated in Figure 19, where v_i is the vertex generated by u_i and v , and coloured vertices in the product are generated by coloured u_1 , u_3 , and u_5 . For brevity, half of the faces in the product are omitted. Consider the automorphism ρ of the product induced by fixing f , f^1 , and f^3 but flipping f^2 (swapping the top and the bottom edges) in two factors. Then ρ fixes the four face blocks on the left and right, and permutes vertices in each of the two middle blocks. In particular, we can permute vertices in a block while its two incident blocks are fixed. Therefore we can permute vertices in one middle block and fix all other five blocks. This gives a non-Cartesian automorphism.

There are two main reasons why we have the above non-Cartesian automorphism. First, there is more than one face block generated by the same faces lying in the same component of the product. Secondly, factors are not rigid enough, so the action on one face block can not affect incident blocks, and can not be transmitted to blocks generated by the same faces. We suspect that if either of these two reasons is absent, then each component of the product might have only Cartesian automorphisms. In particular, if the 1-skeleton of each factor is bipartite, then face blocks generated by the same faces are in different components. Also note that if a complex is a surface, it is rigid enough that the action on one face completely determines the whole automorphism. So far we do not have a definite result yet, and hence we pose the following two conjectures. We hope to resolve these problems in the near future.

Conjecture 5.11. For $i \in \{1, 2, \dots, m\}$, suppose that X_i is an elementary ordinary complex with faces of the same even length $2n \geq 6$, and X_i has bipartite 1-skeleton. Then for any component X of the complex tensor product $\otimes_{i=1}^m X_i$, $\text{Aut}(X)$ can be generated by automorphisms of X_i 's together with permutations of isomorphic factors.

Conjecture 5.12. For $i \in \{1, 2, \dots, m\}$, suppose that X_i is an elementary ordinary complex with faces of the same even length $2n \geq 6$, and X_i has surface structure. Then

for any component X of the complex tensor product $\otimes_{i=1}^m X_i$, $\text{Aut}(X)$ can be generated by automorphisms of X_i 's together with permutations of isomorphic factors.

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